Minkowski content for the scaling limit of the 3D LERW

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Joint work with
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Statistical Mechanics Beyond 2D
IPAM, UCLA
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Loop-erased random walk (LERW)

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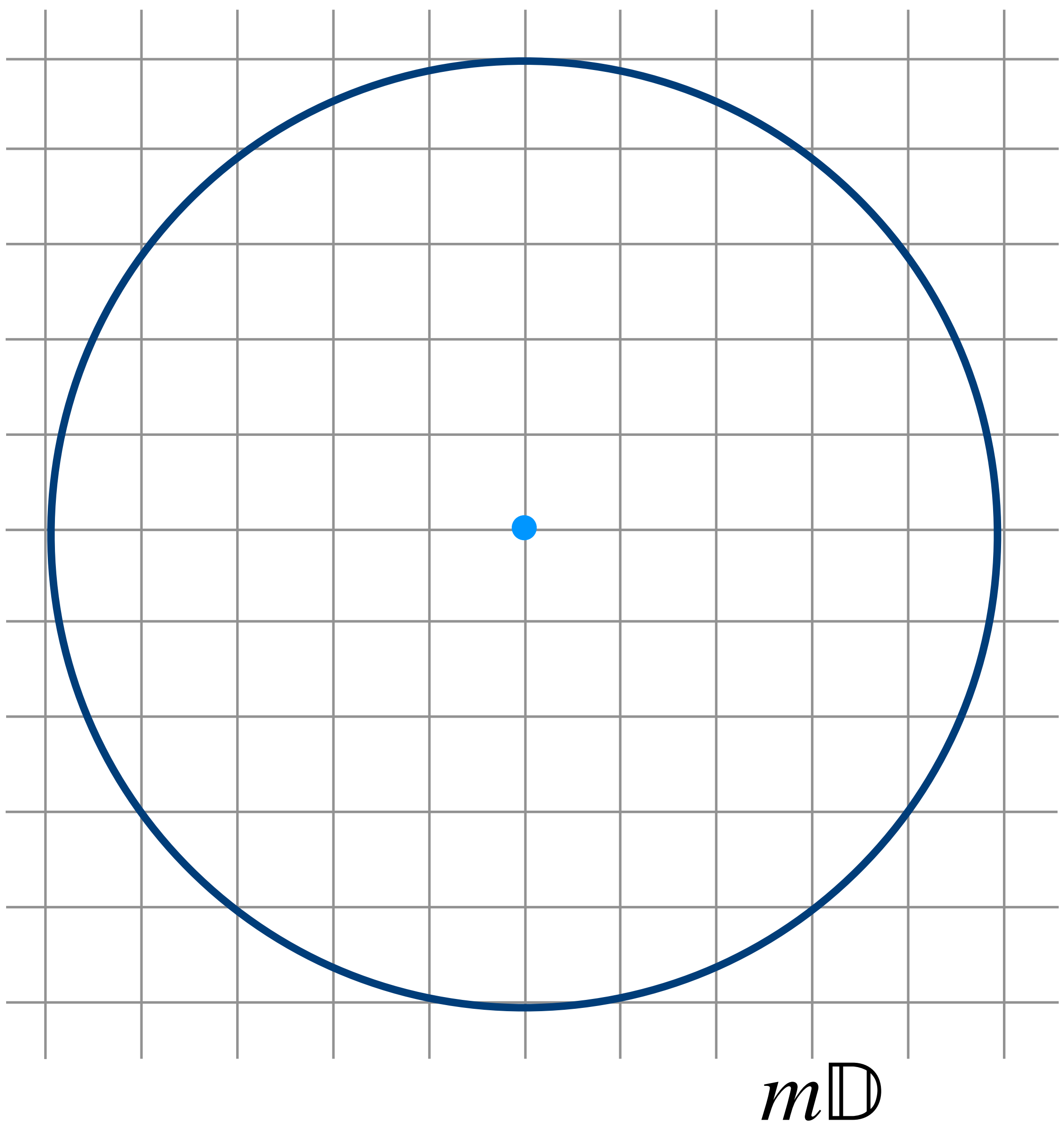
- $S[0,\tau_m] = [S_0, \ldots, S_{\tau_m}]$



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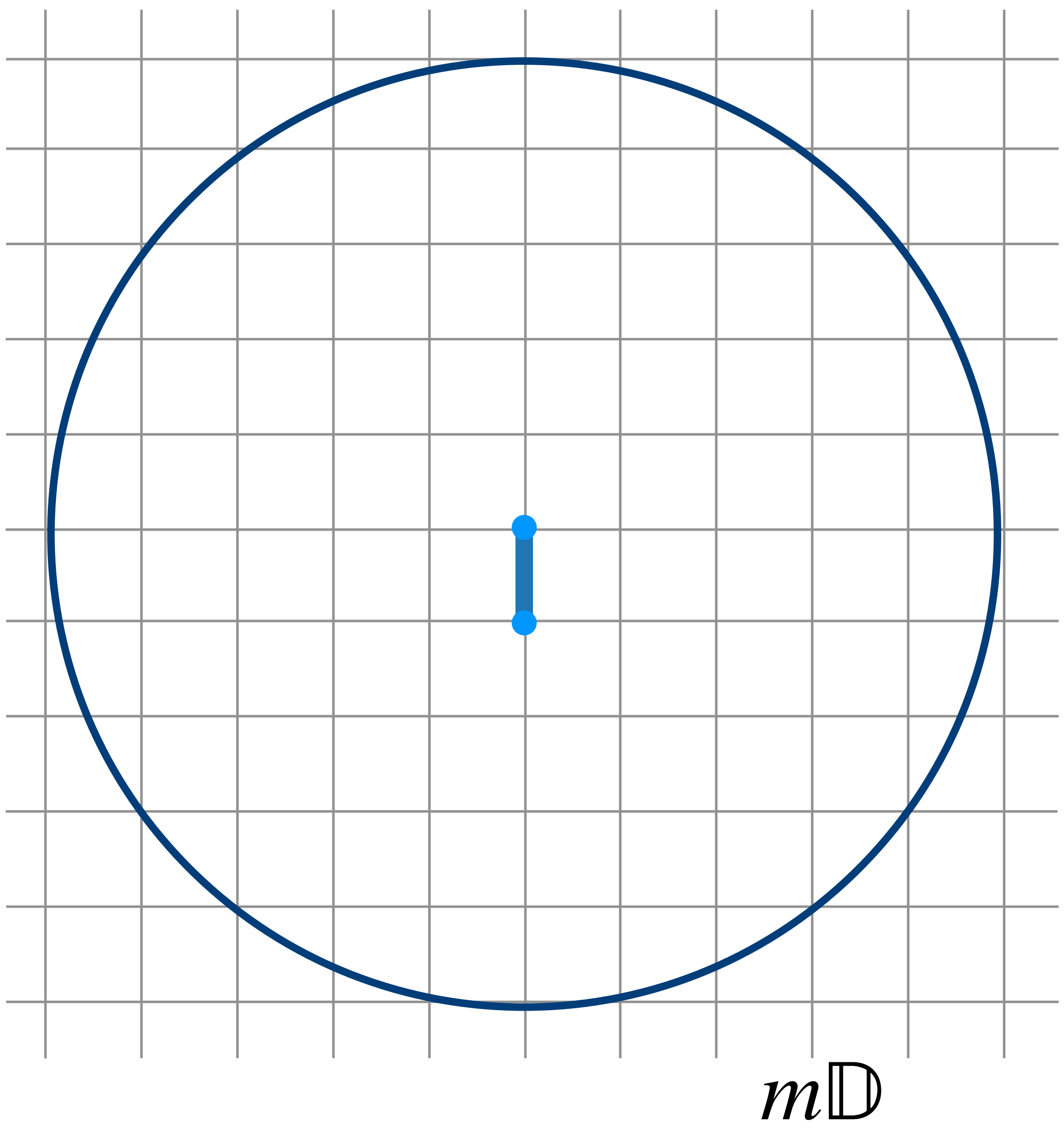

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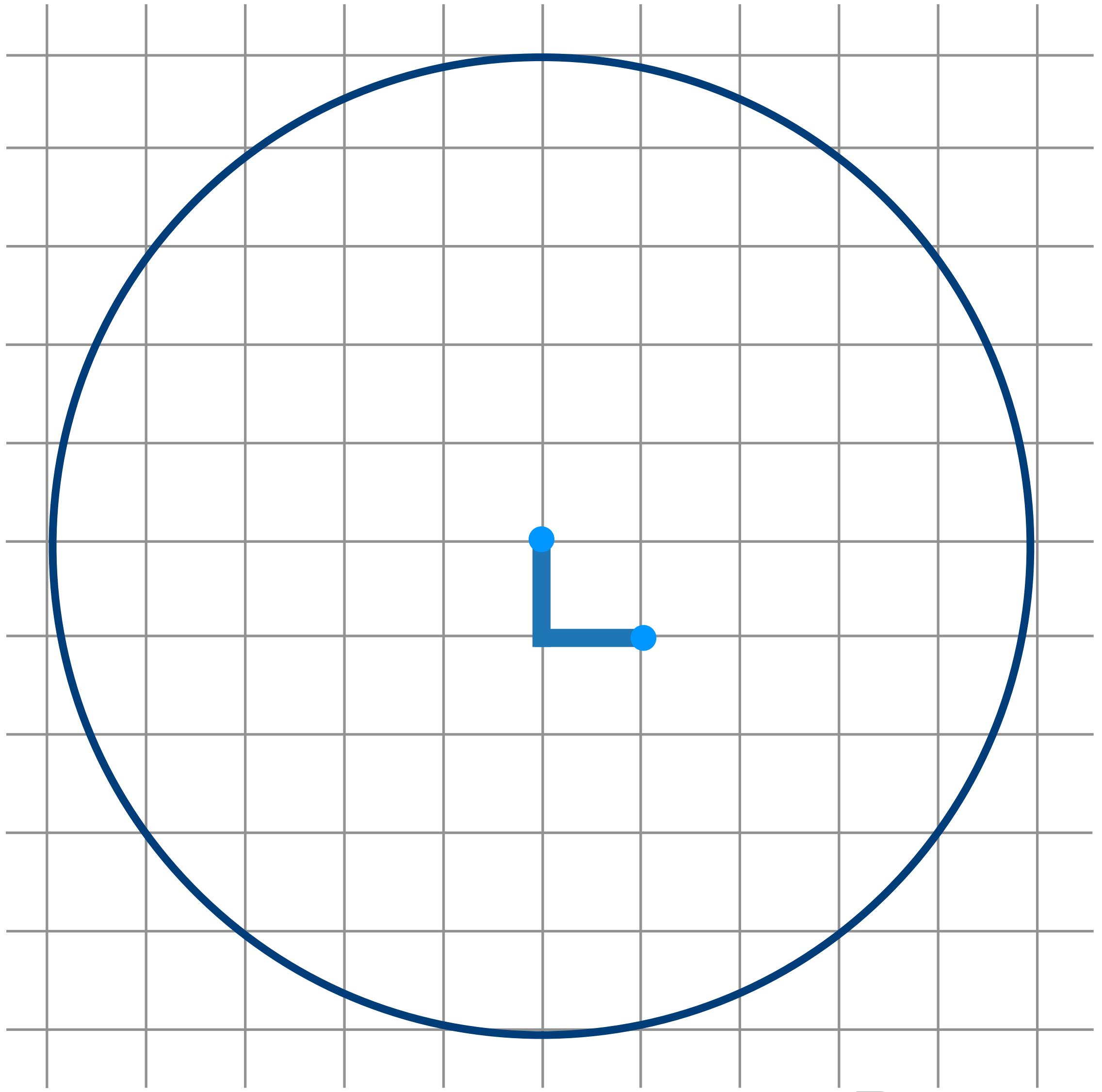

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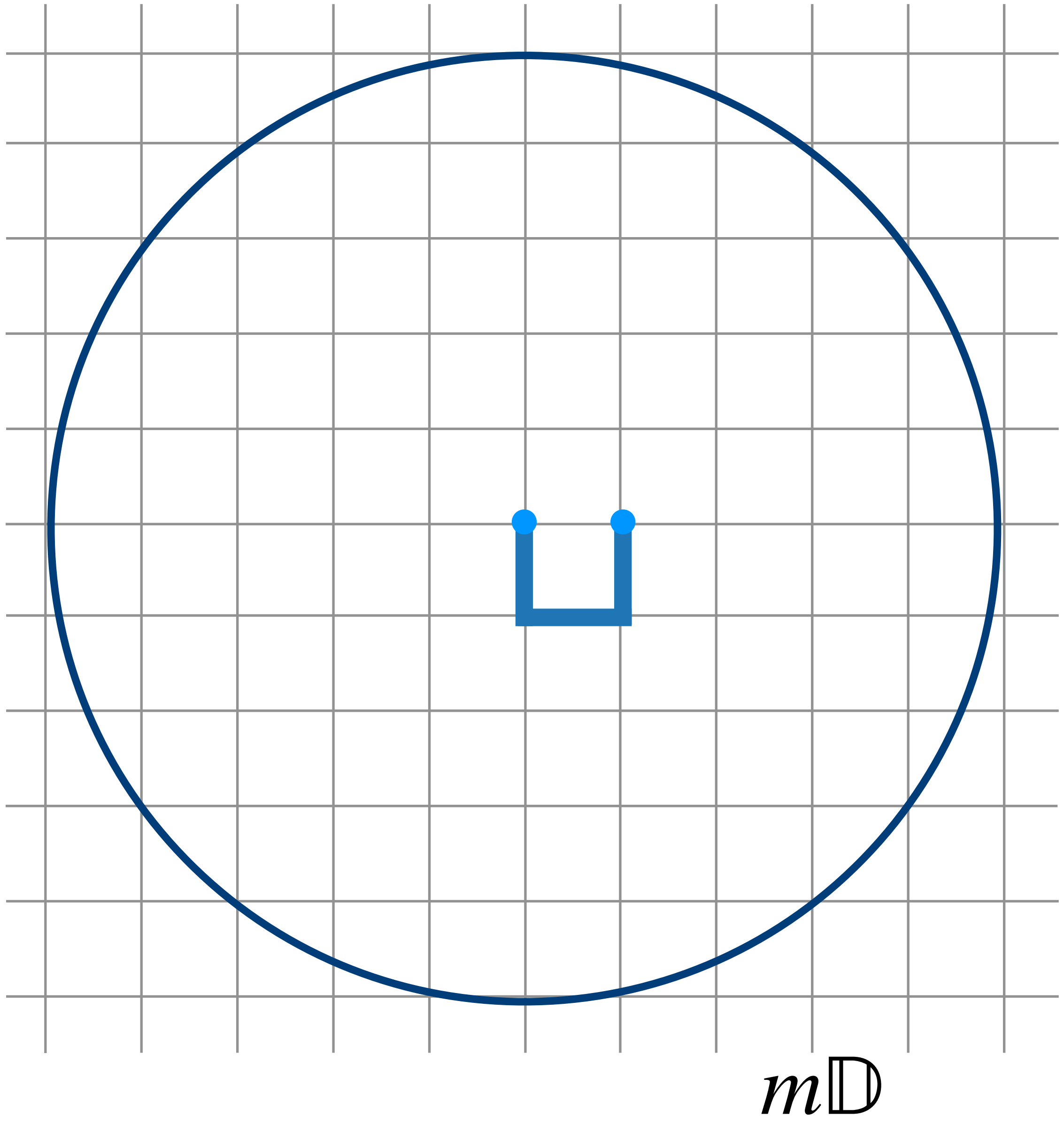

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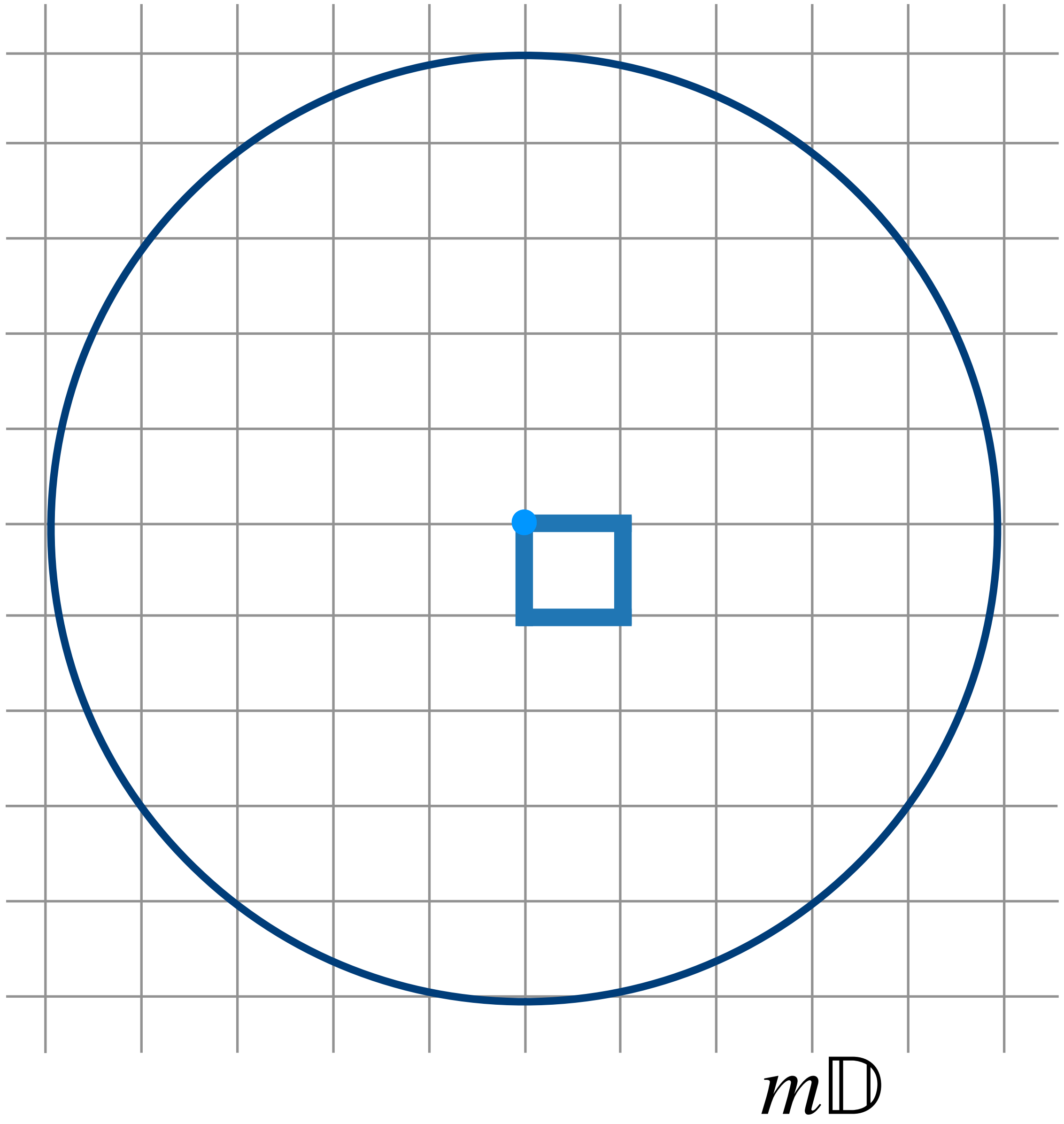

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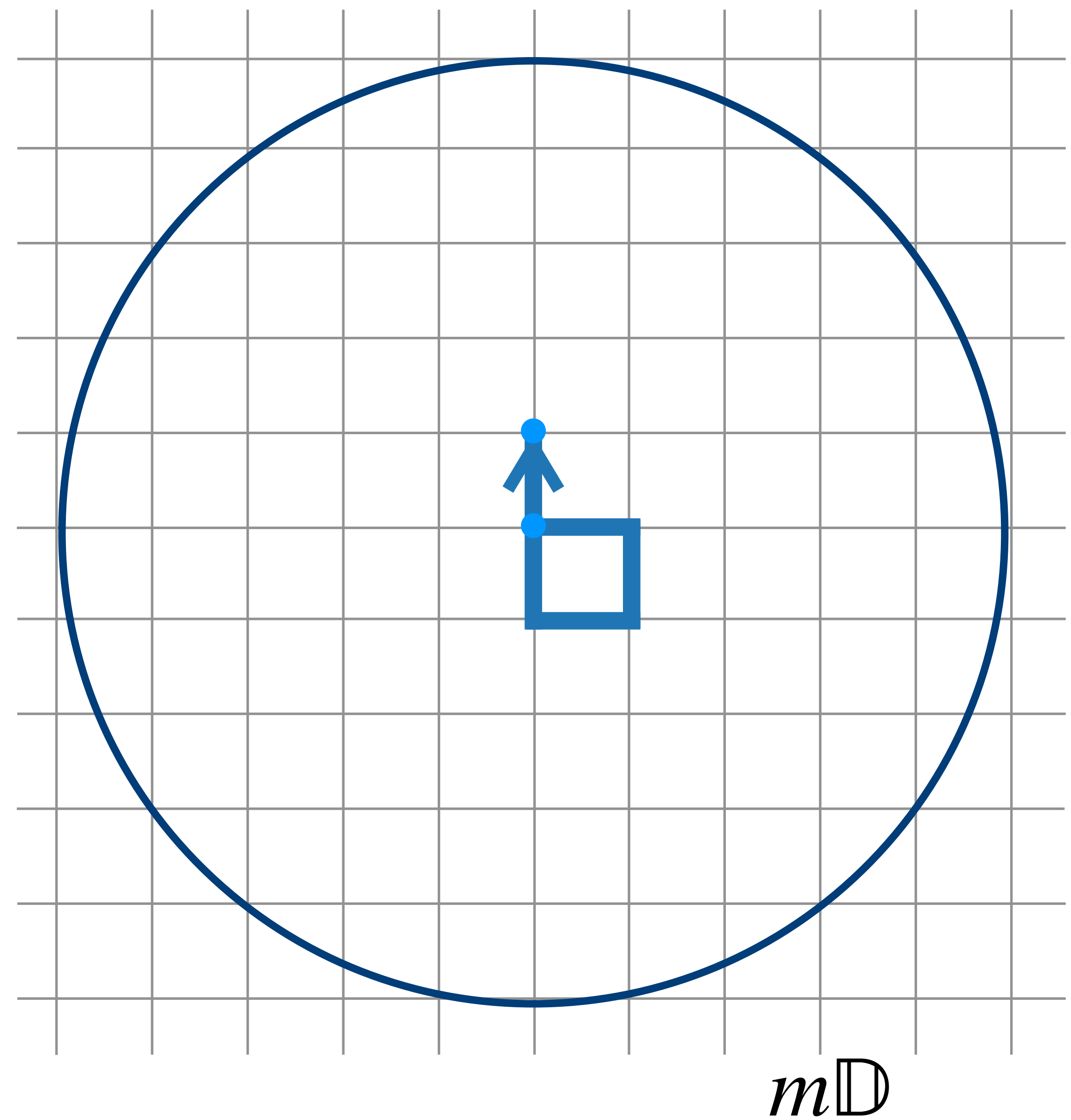

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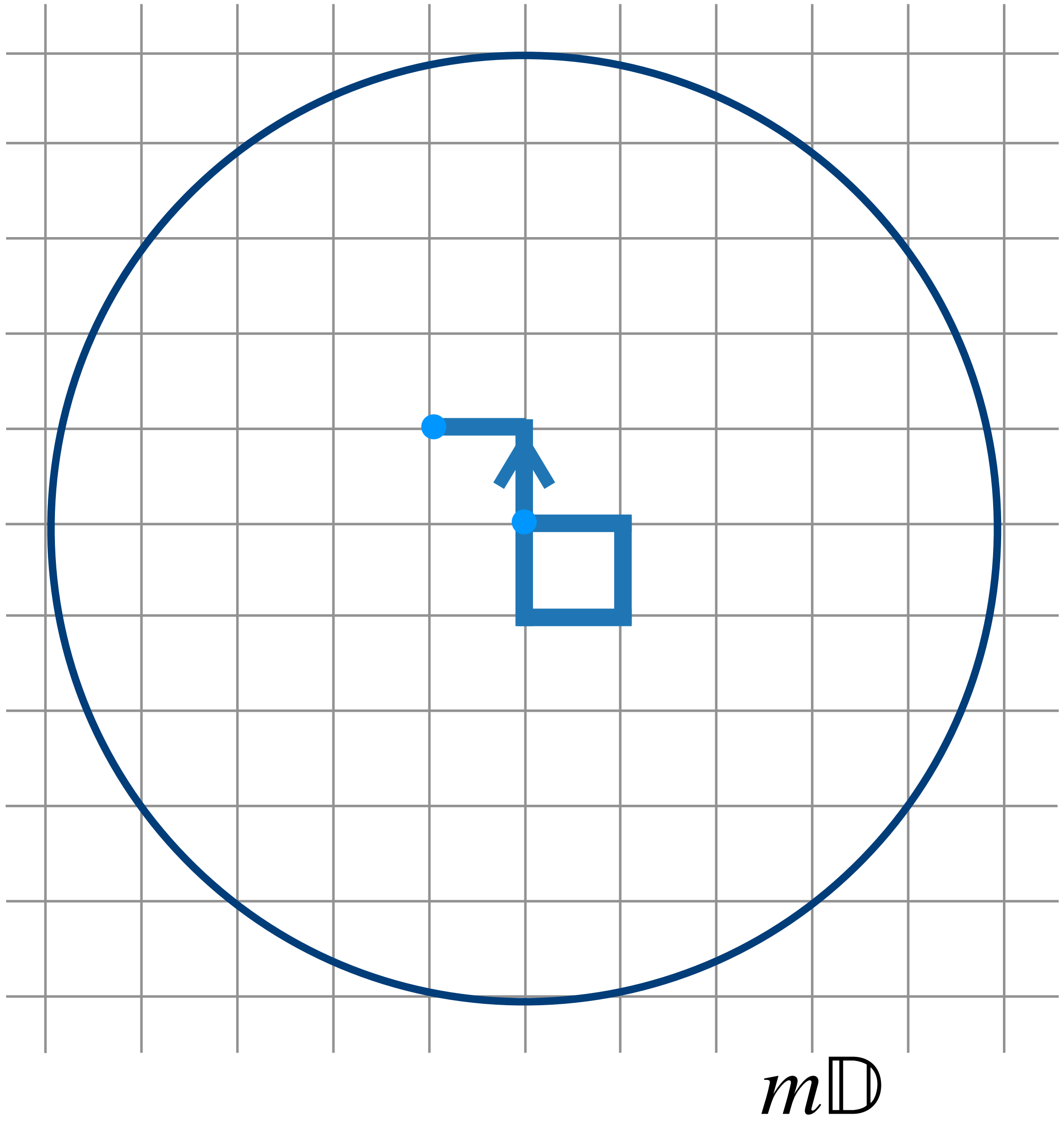

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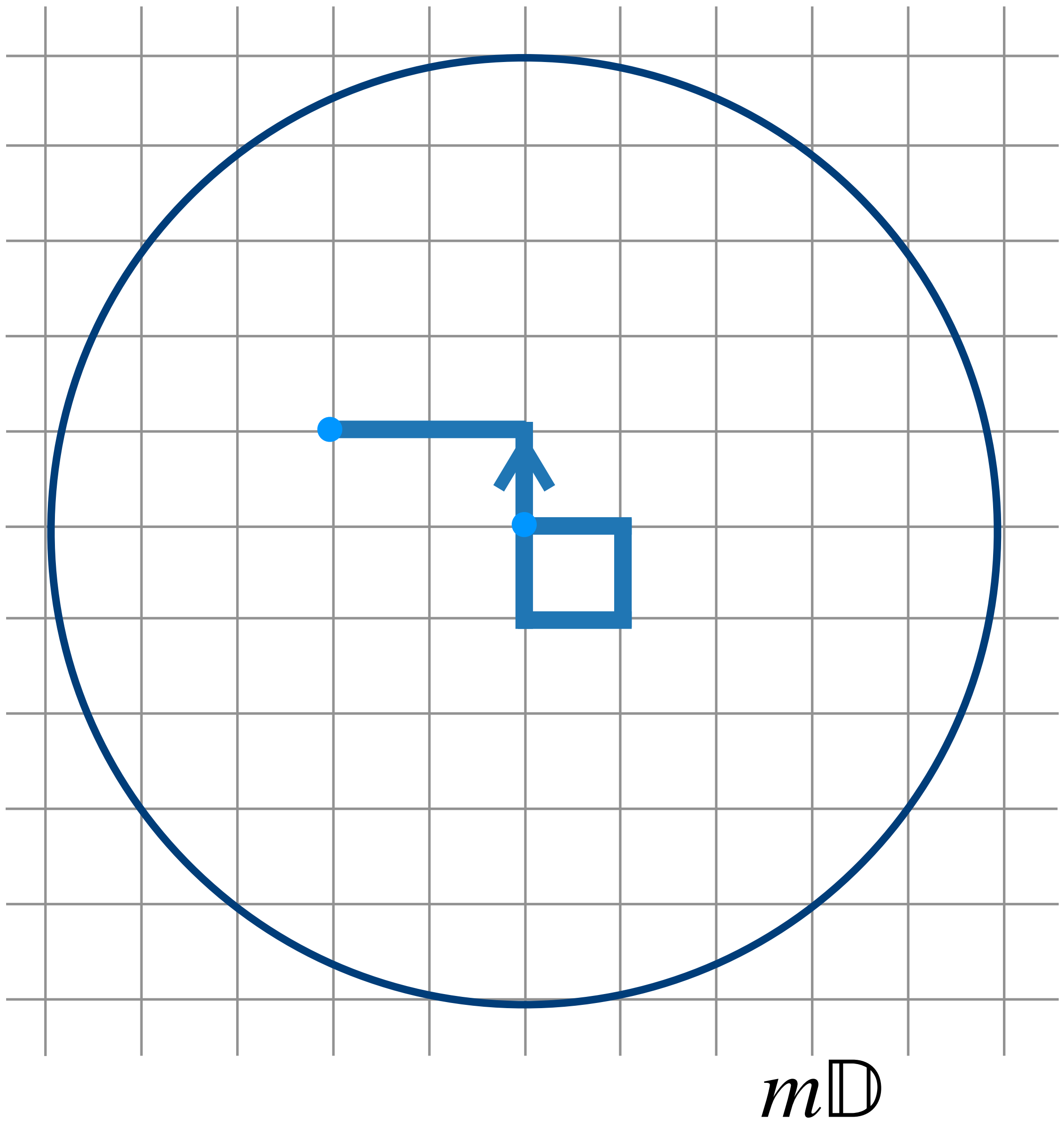

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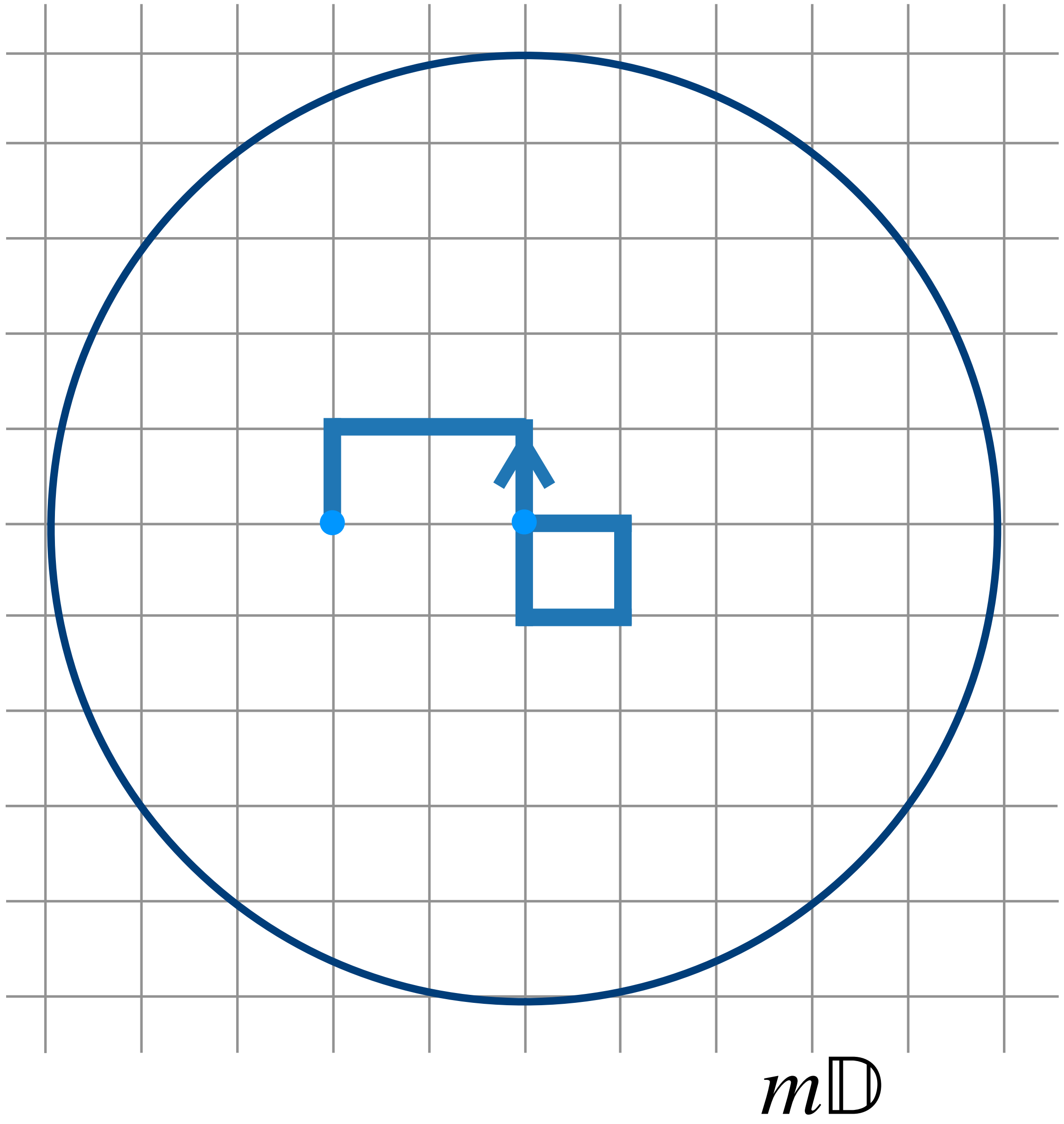

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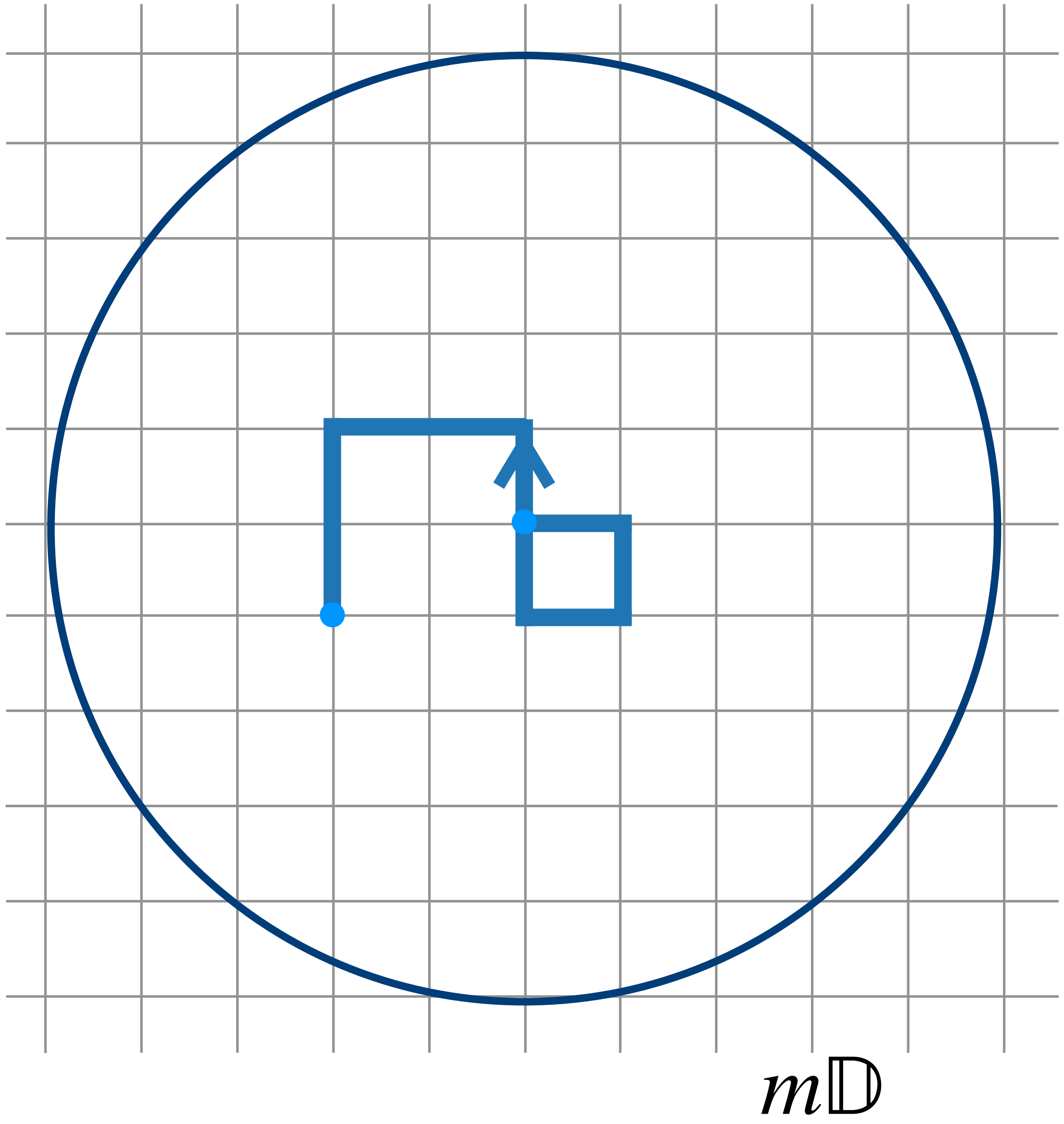

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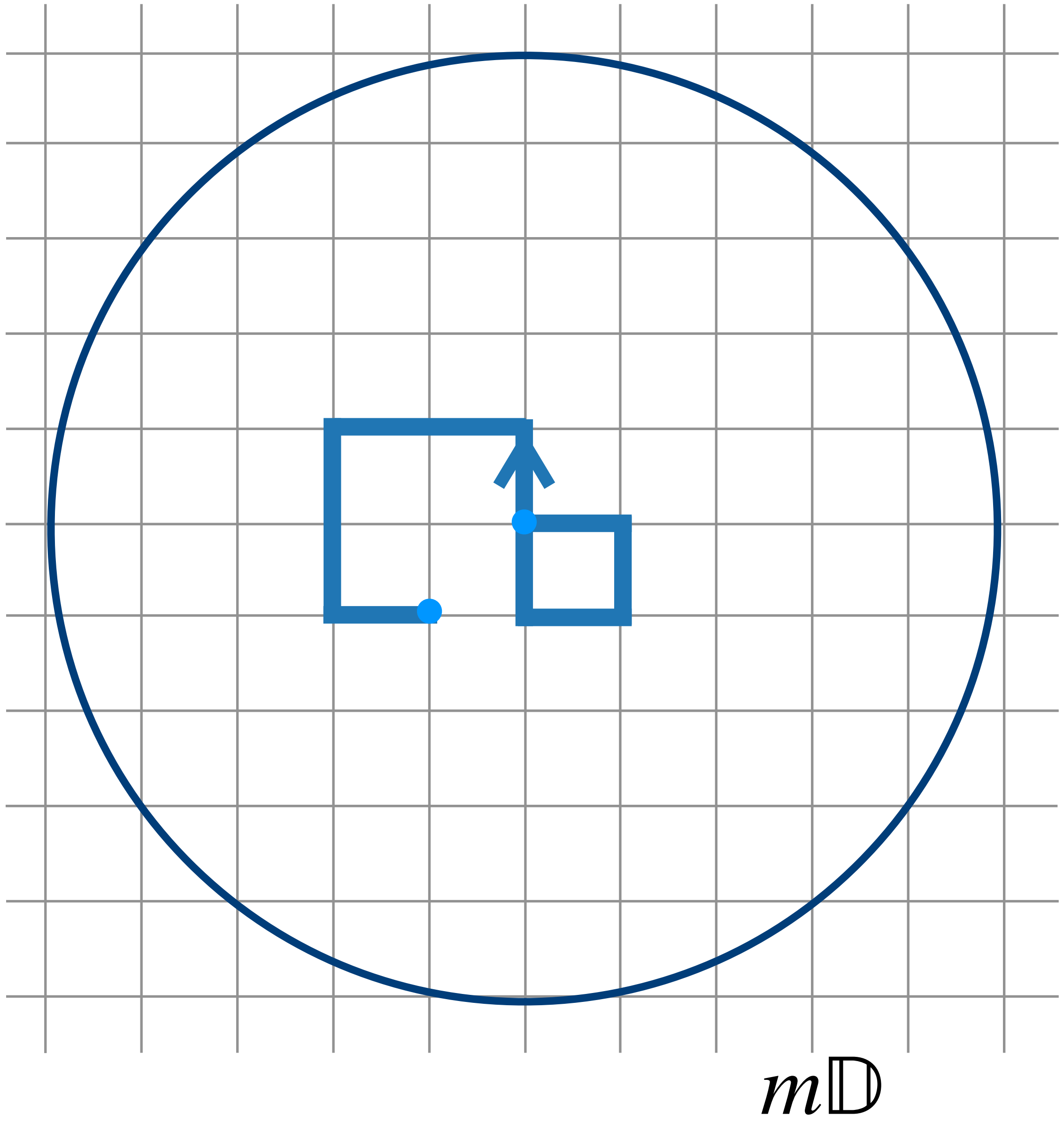

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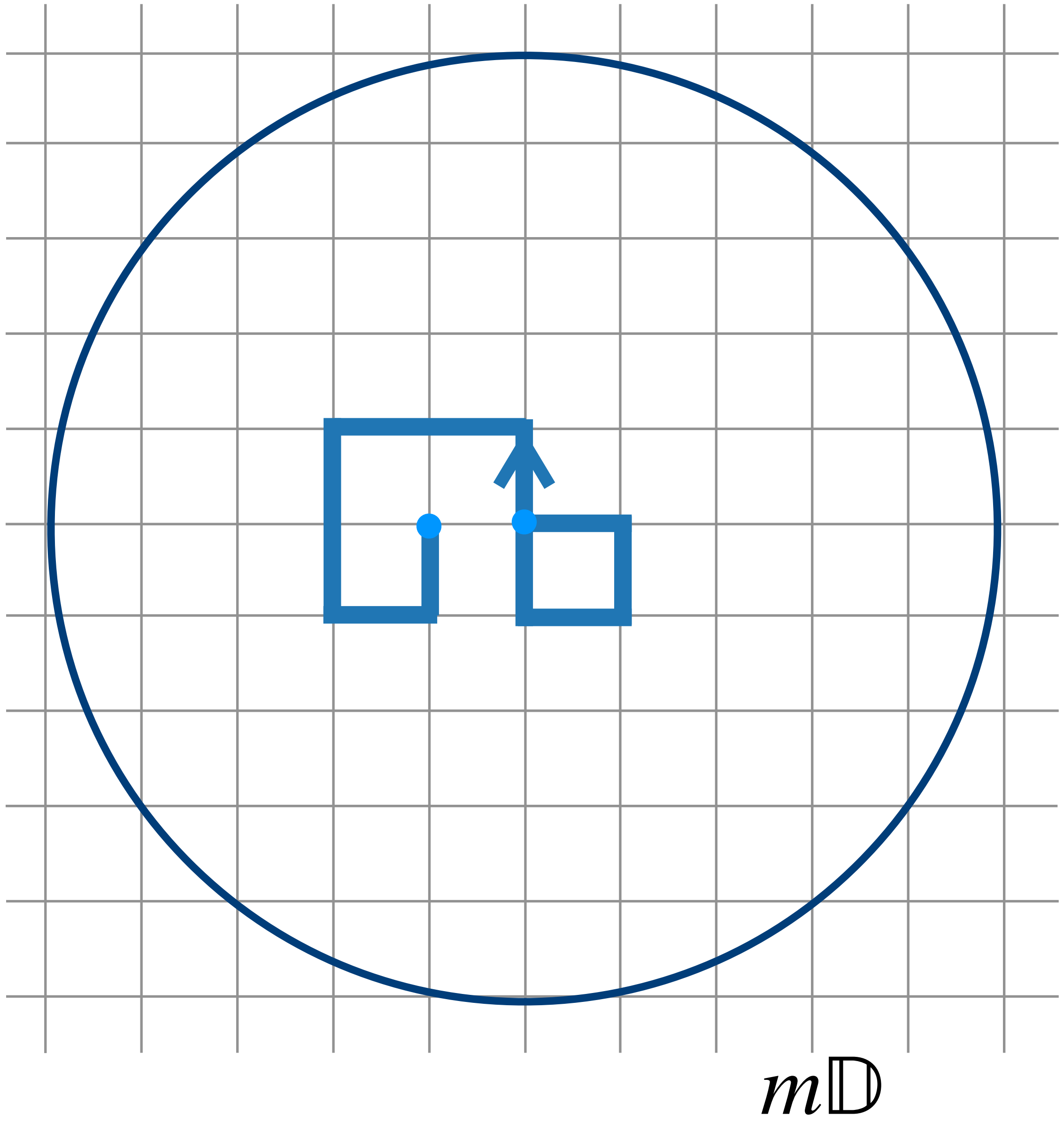

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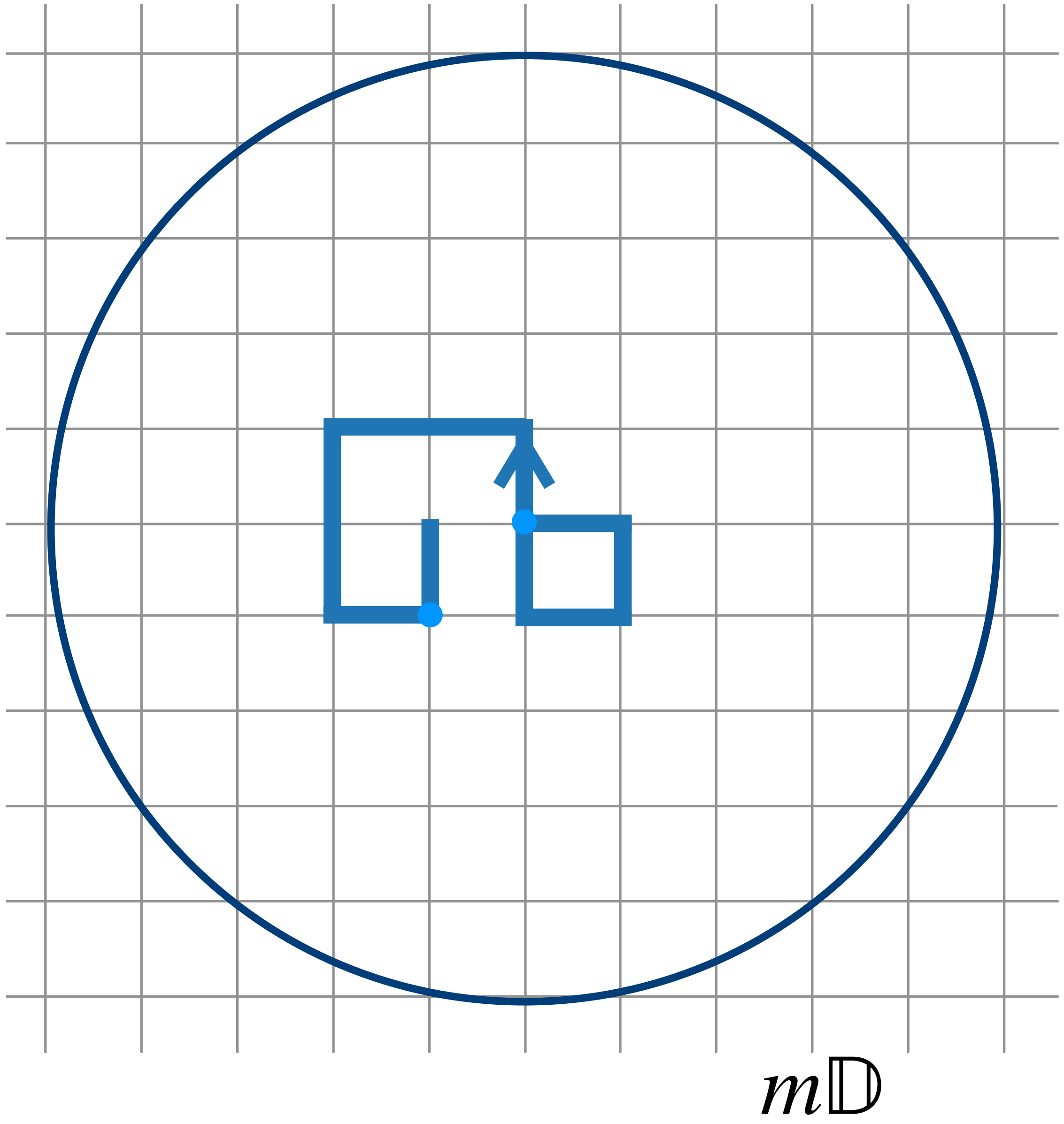

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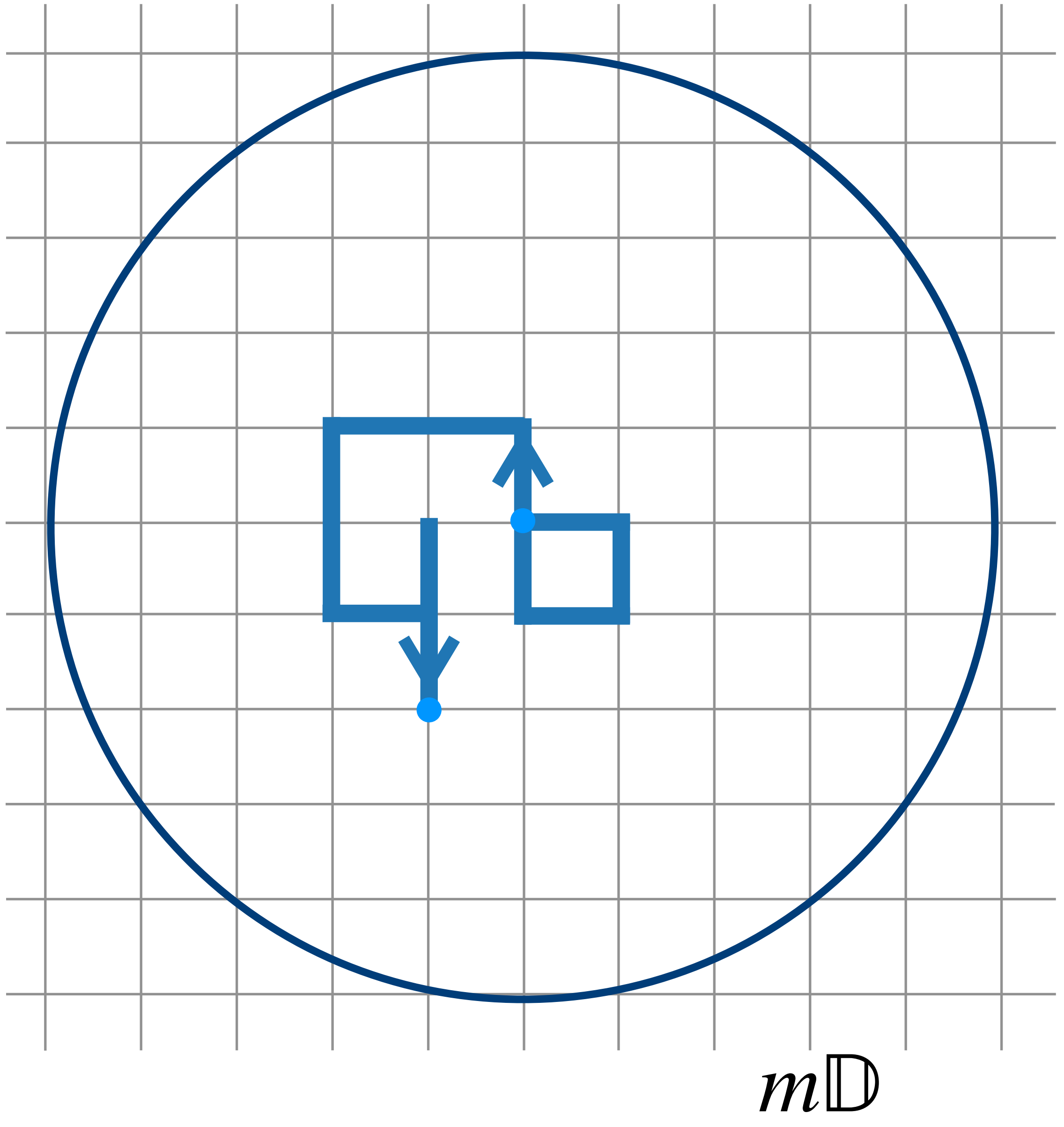

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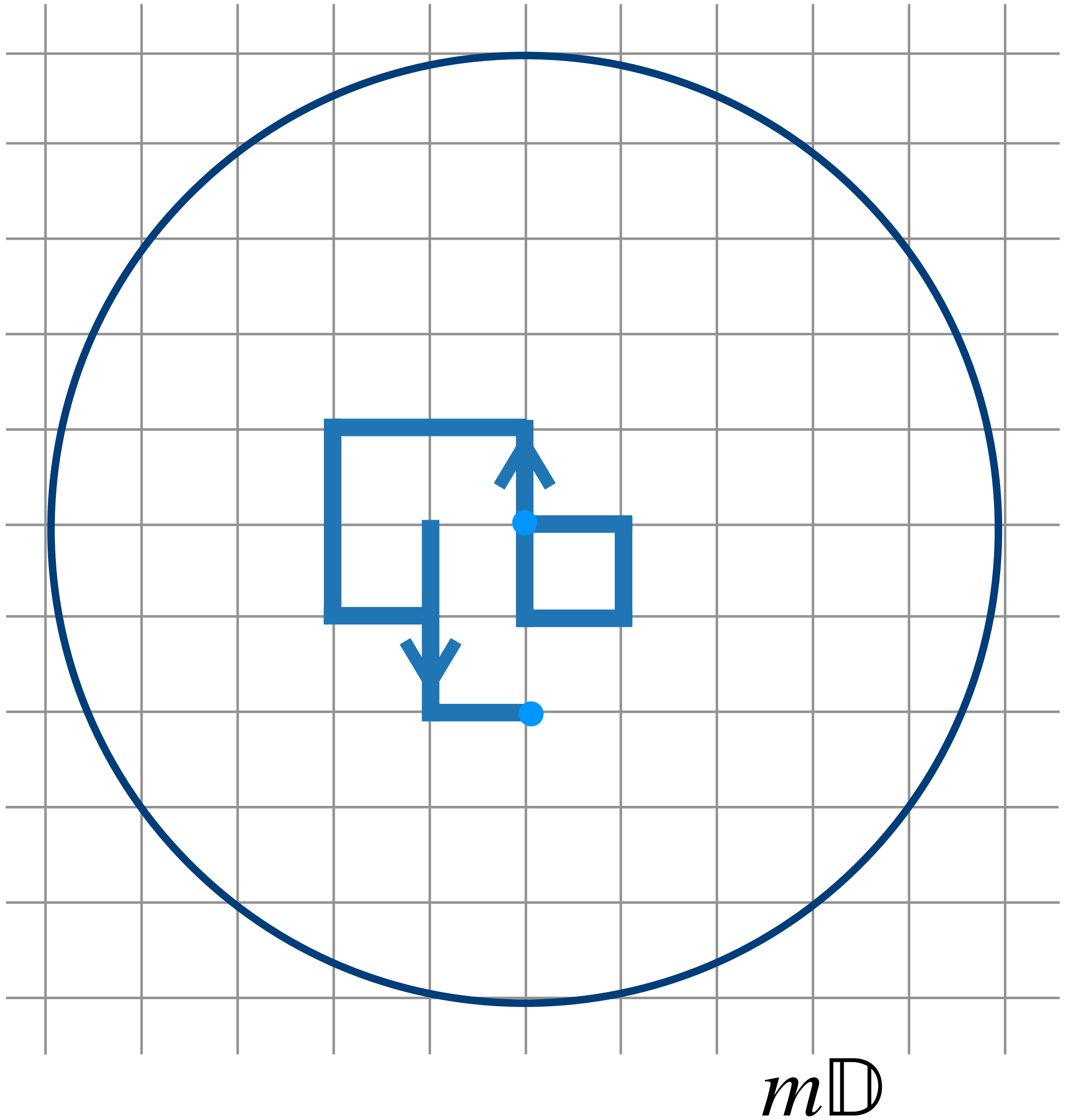

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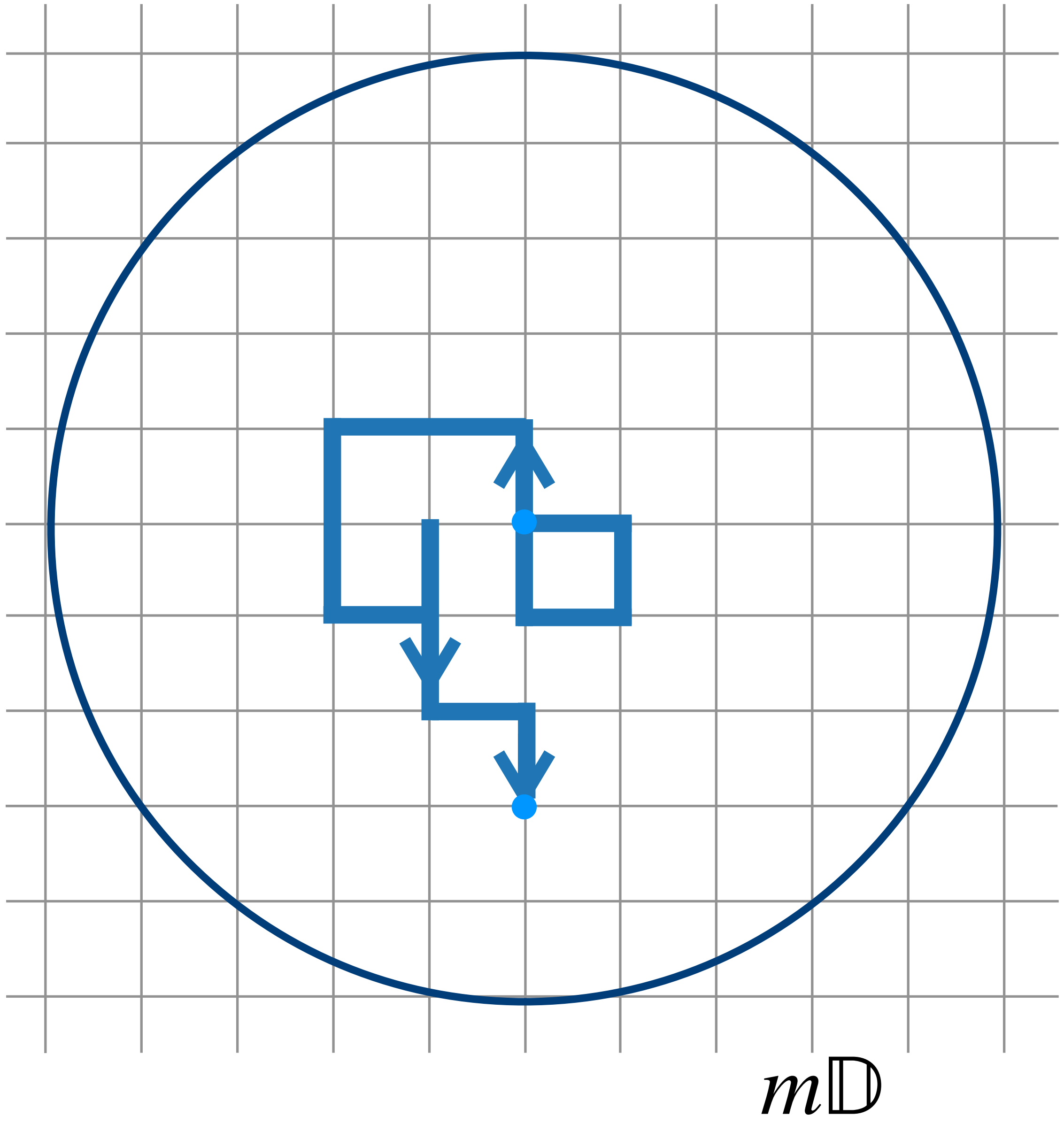

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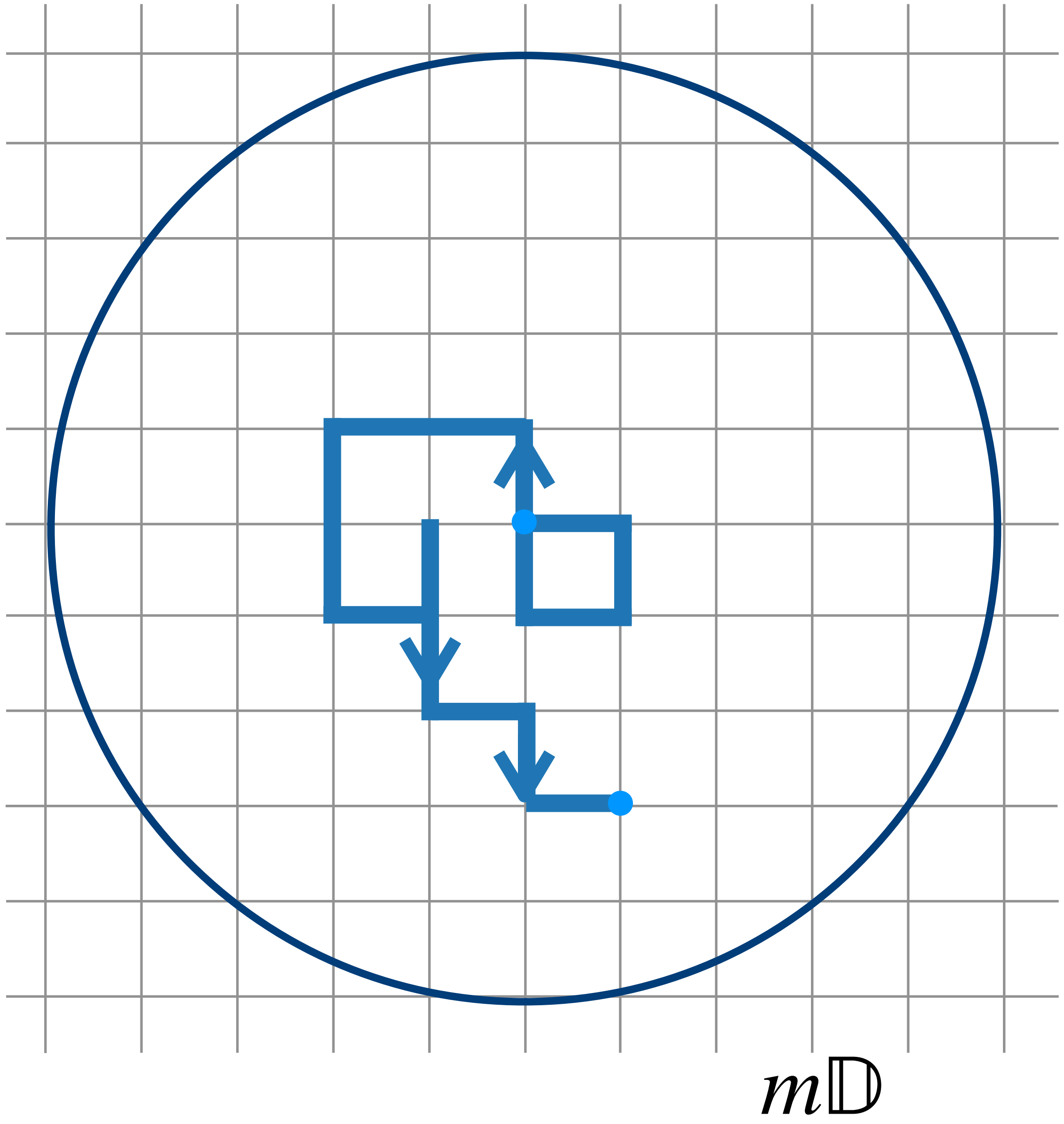

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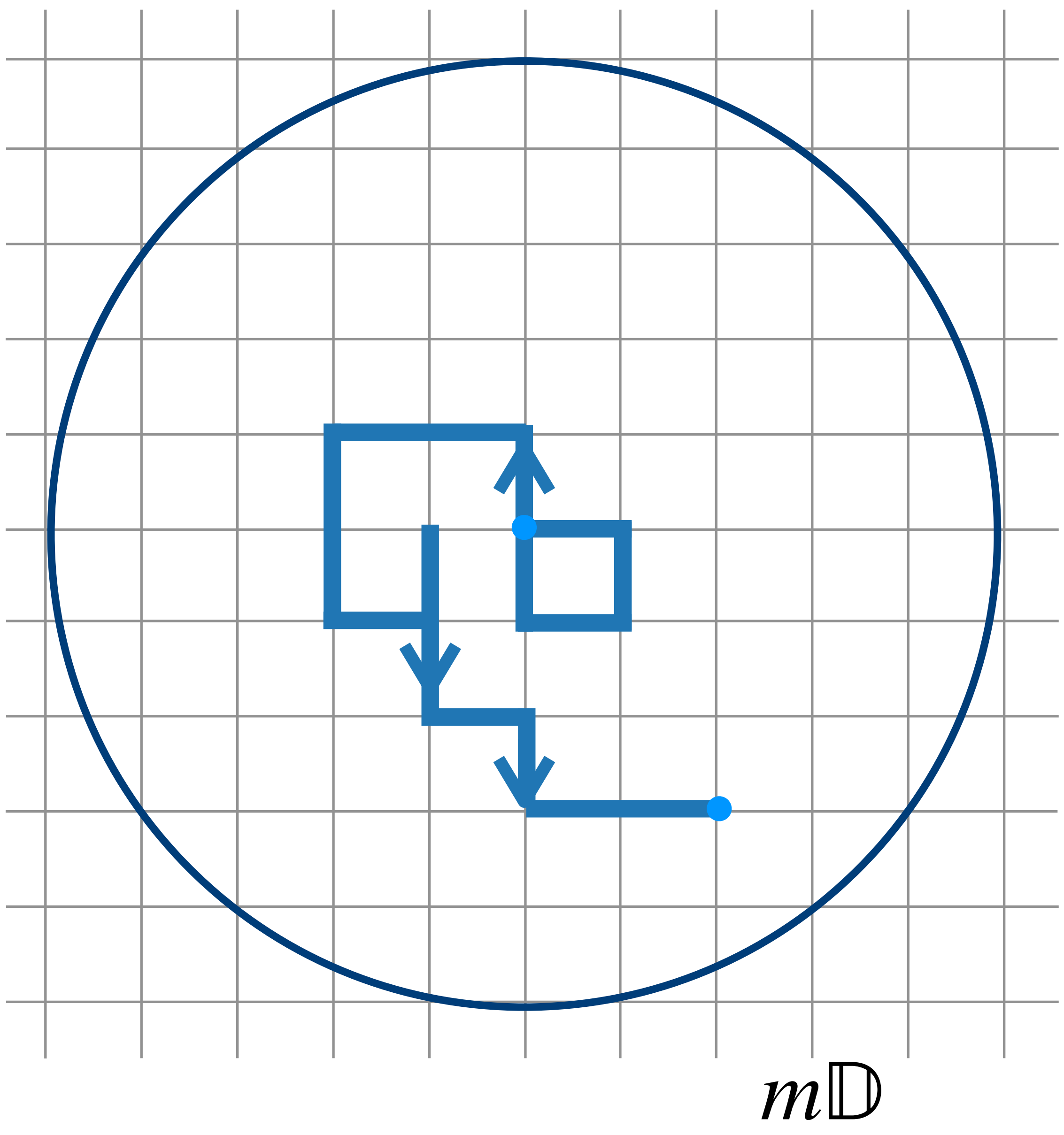

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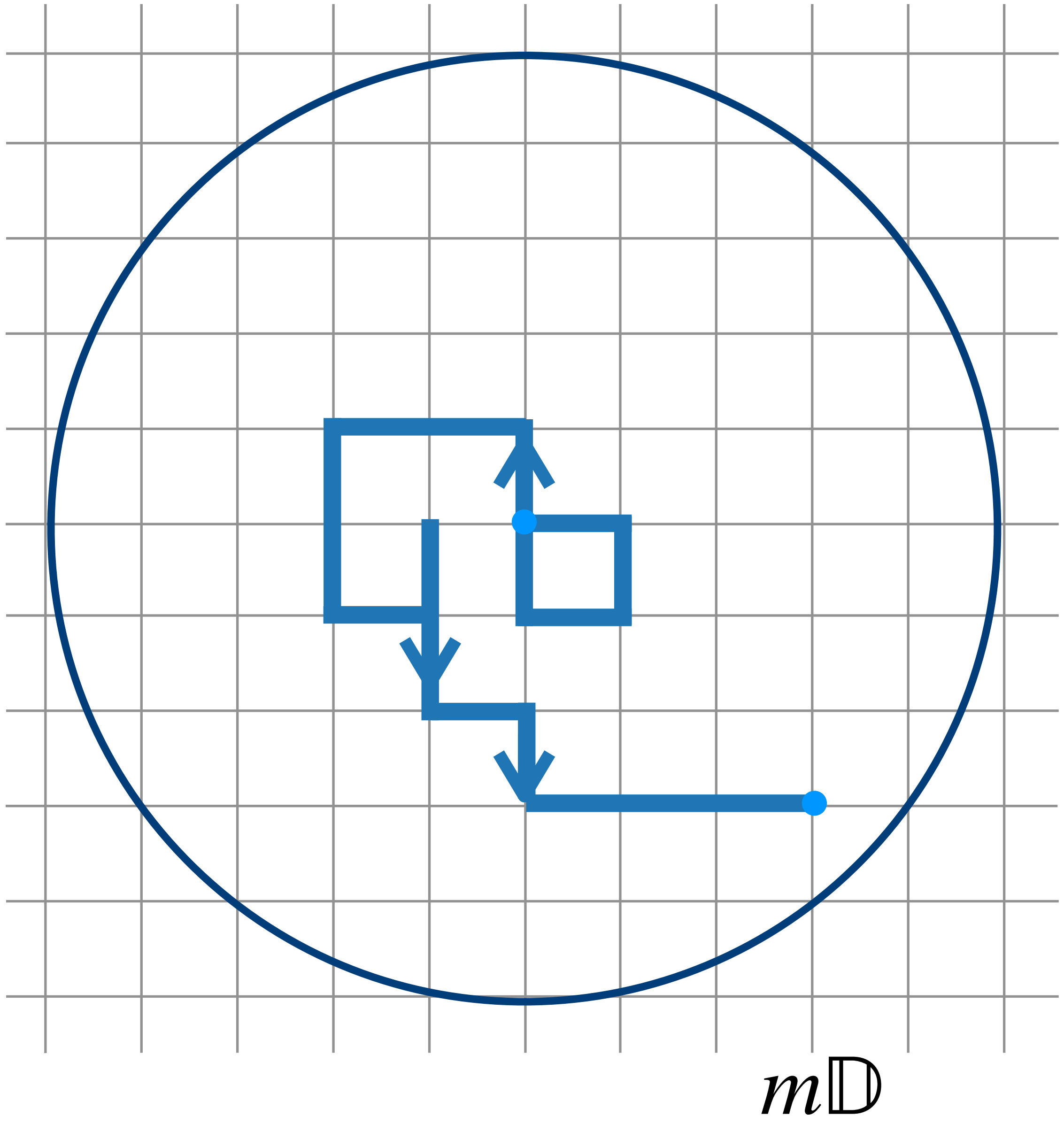

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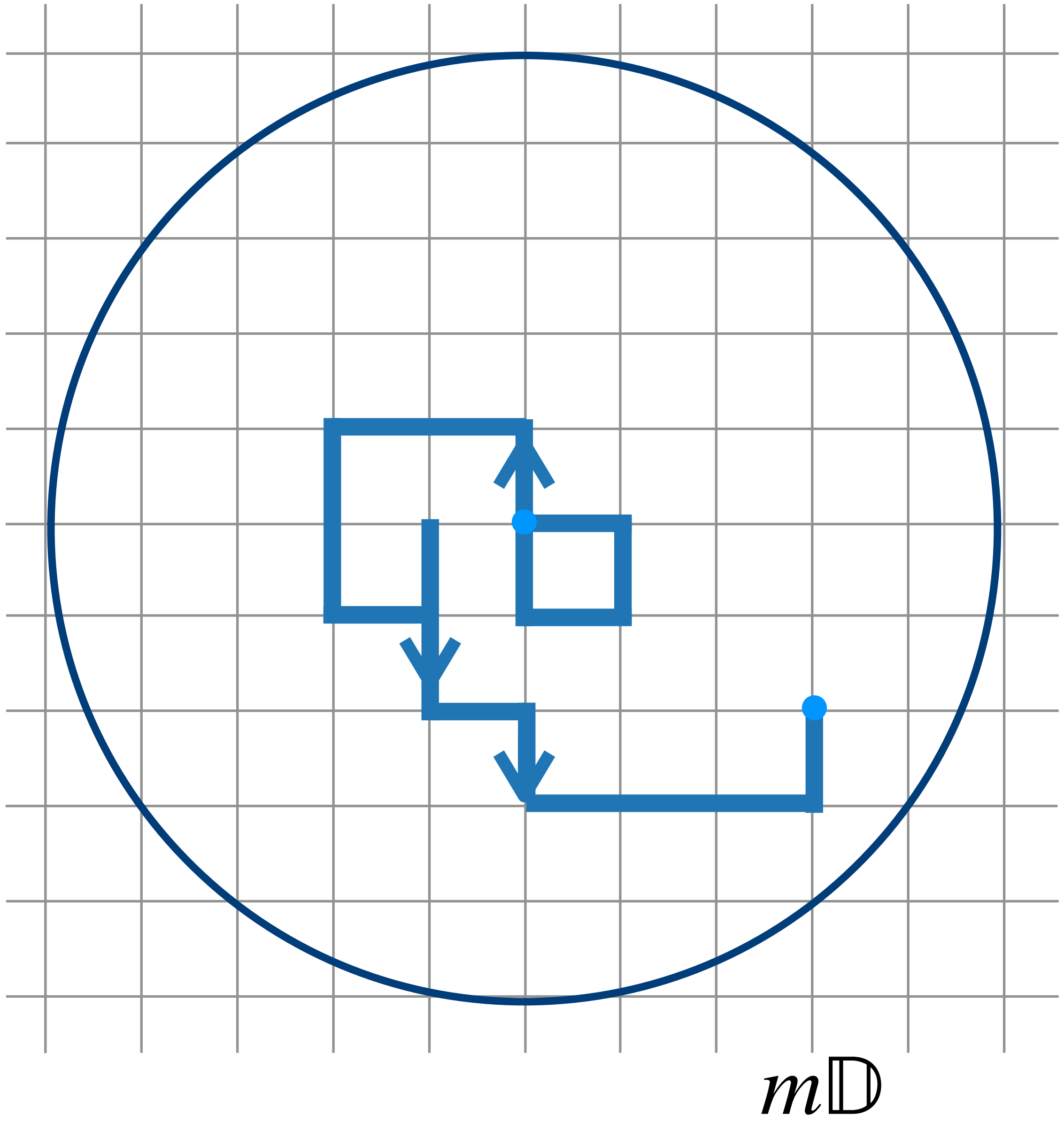

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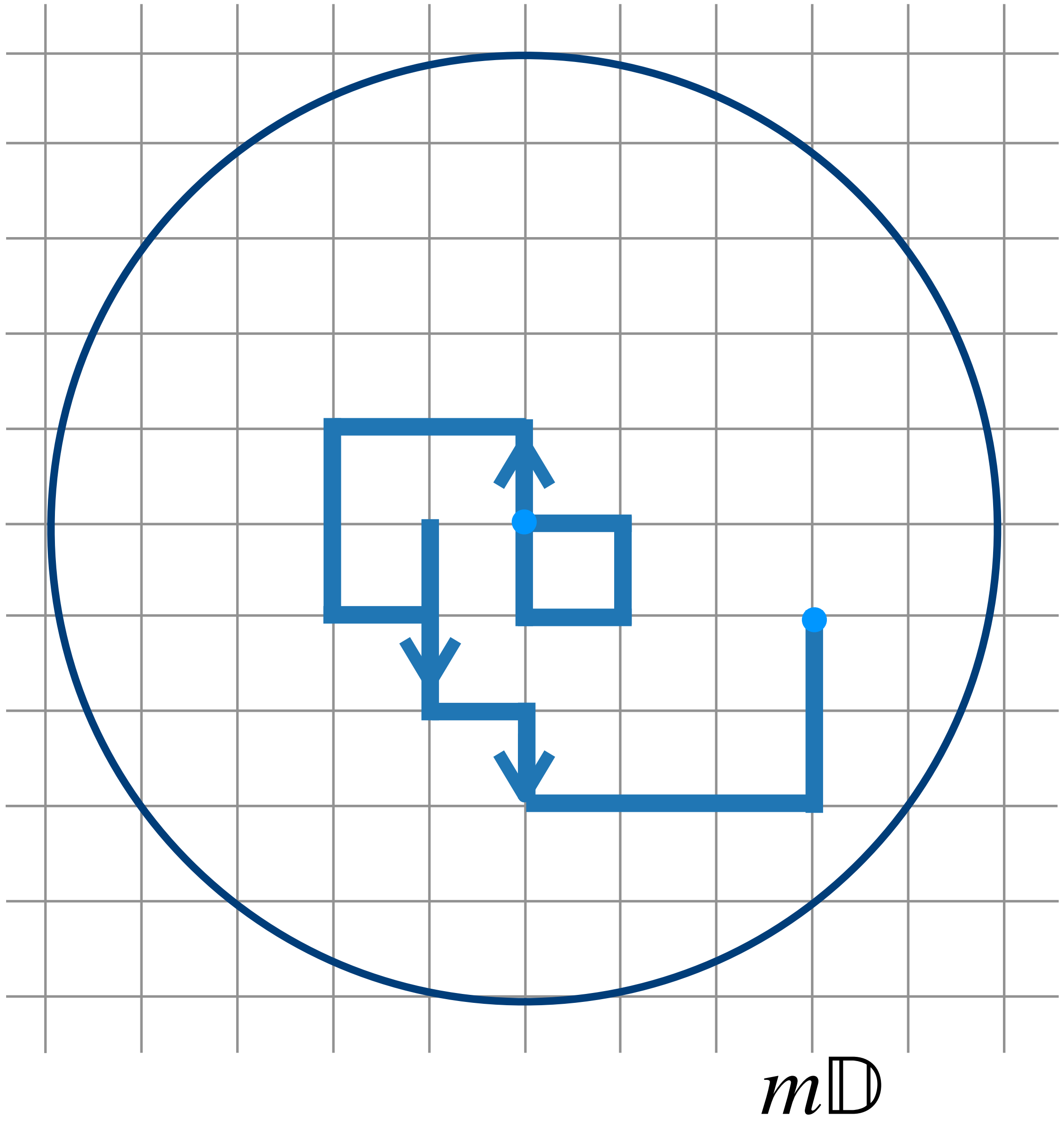

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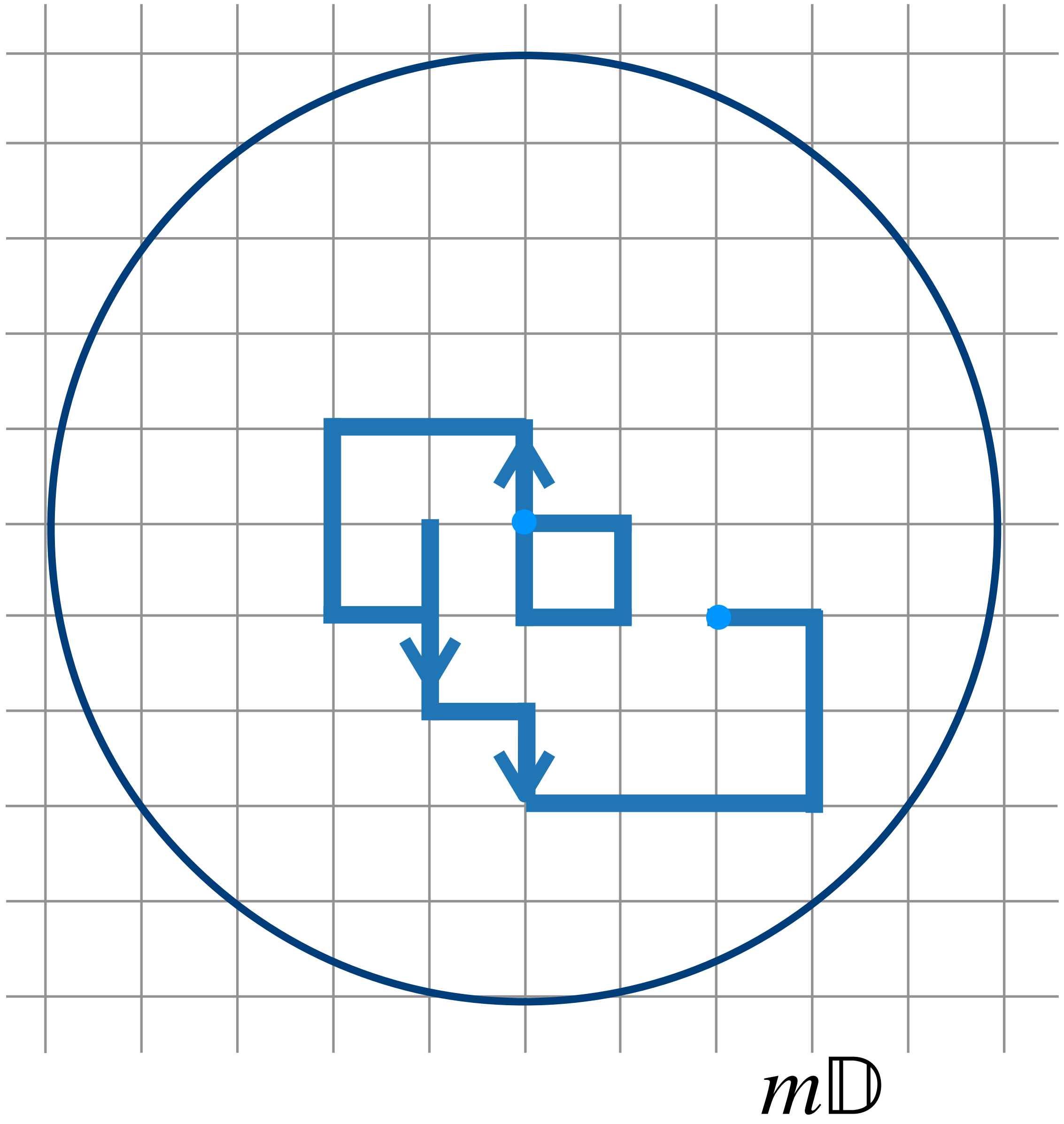

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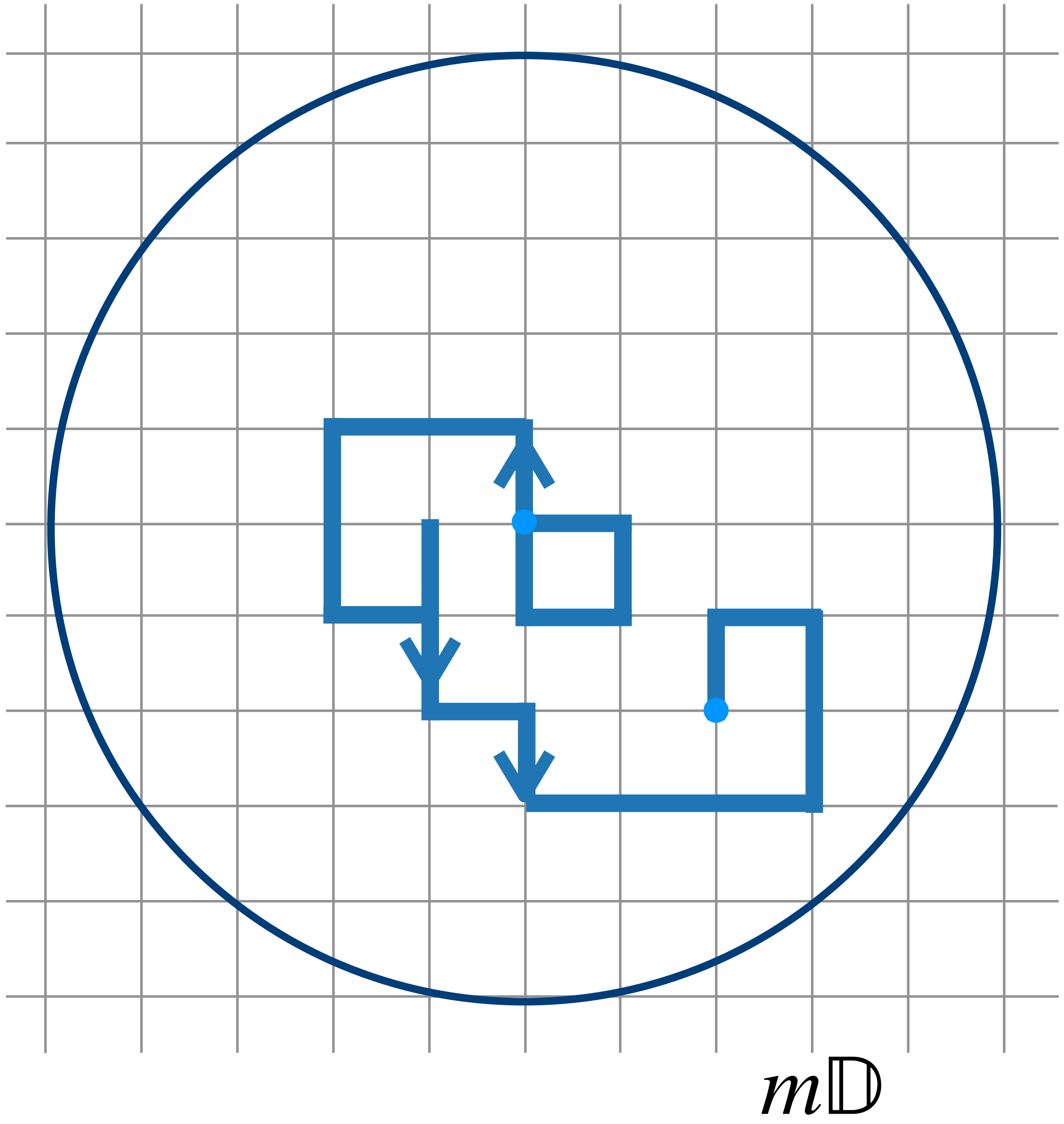

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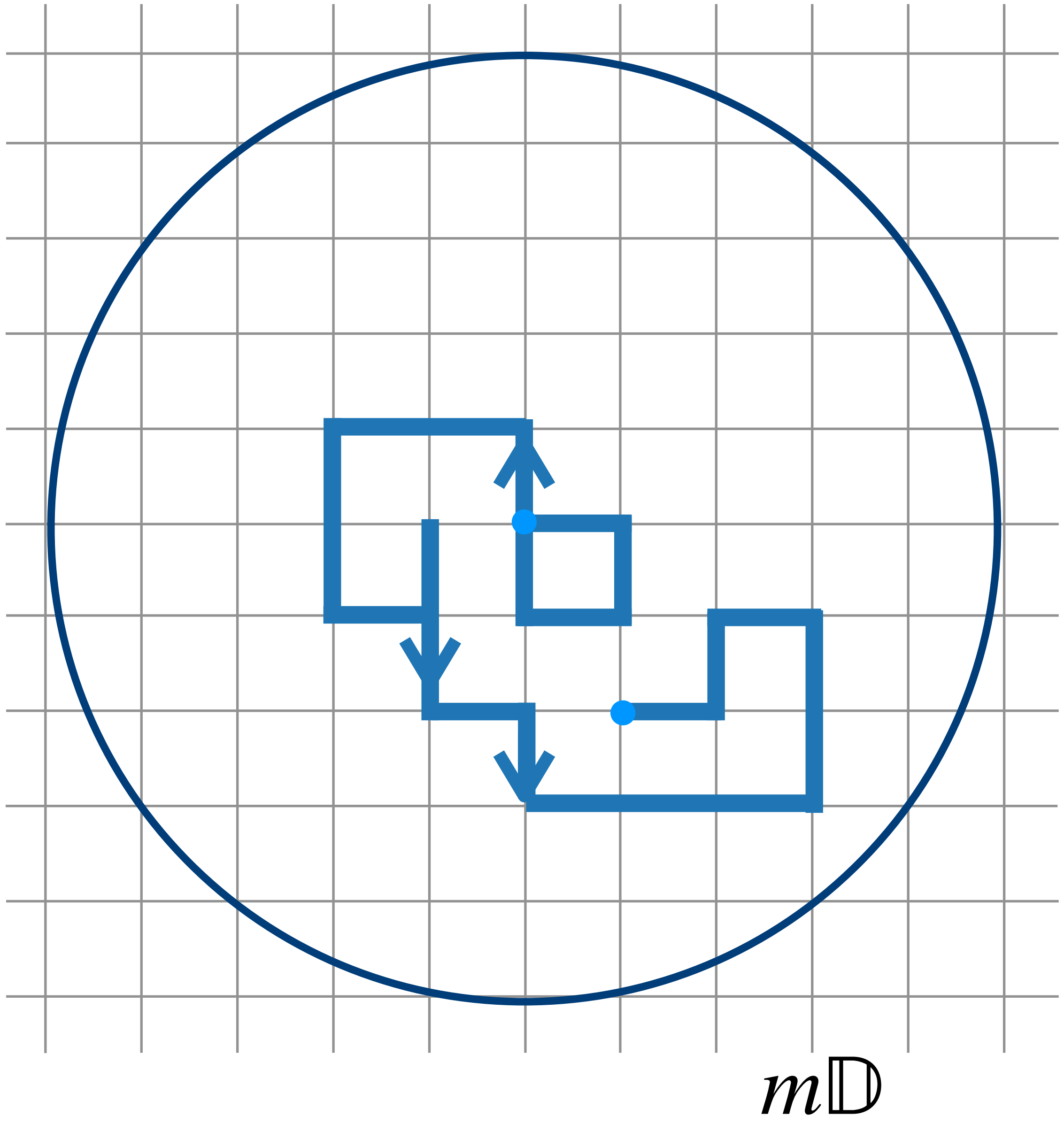

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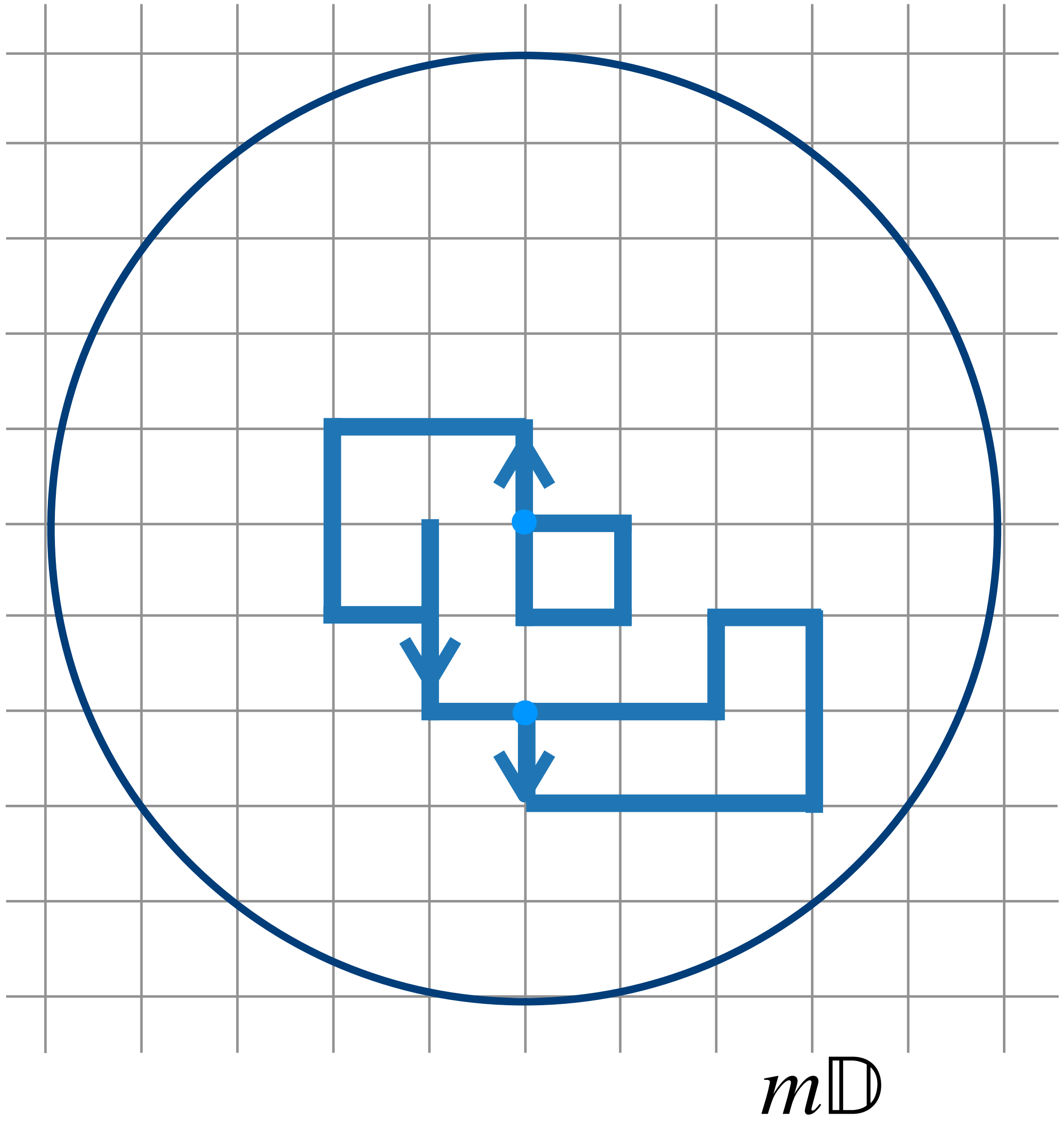

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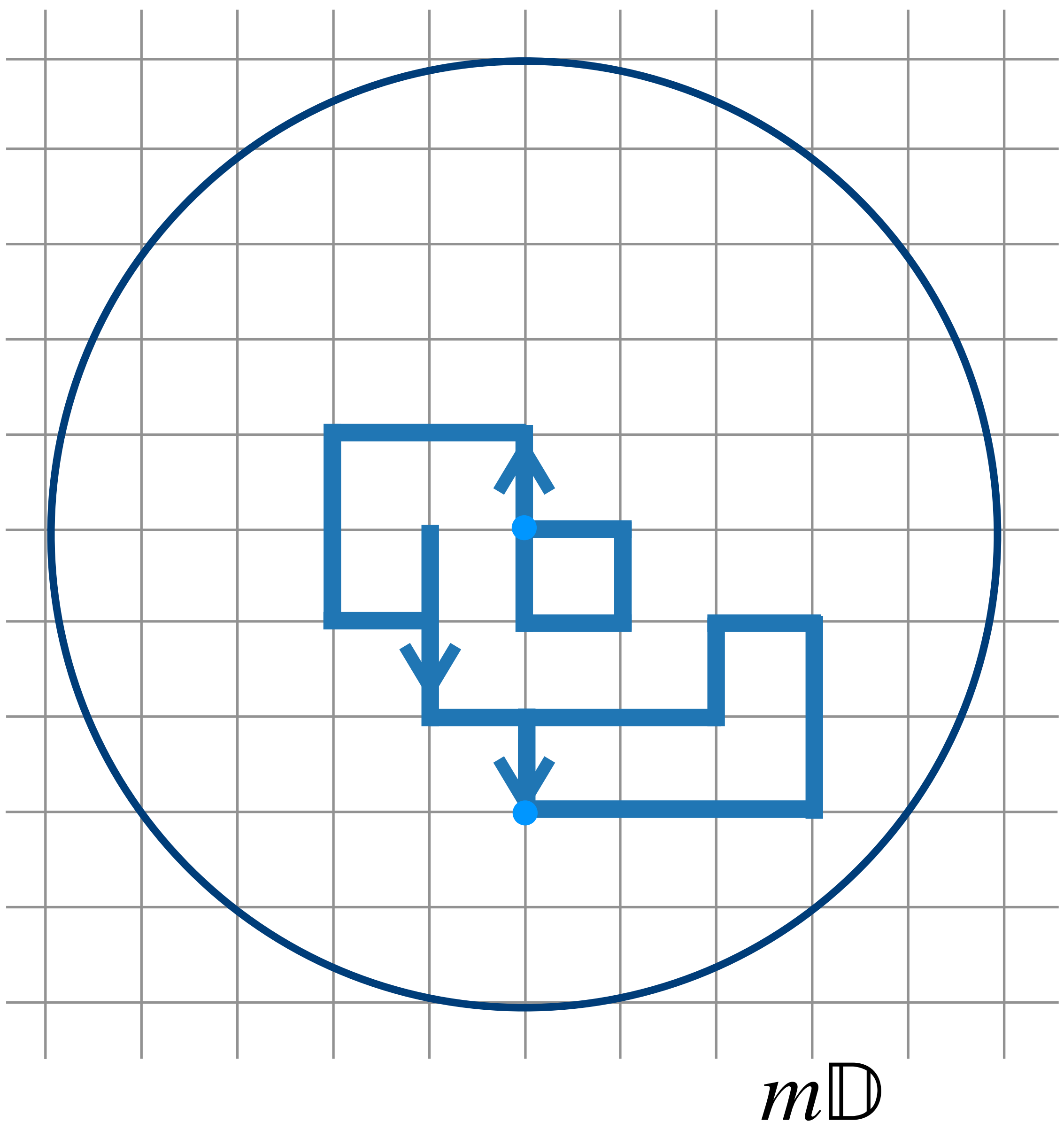

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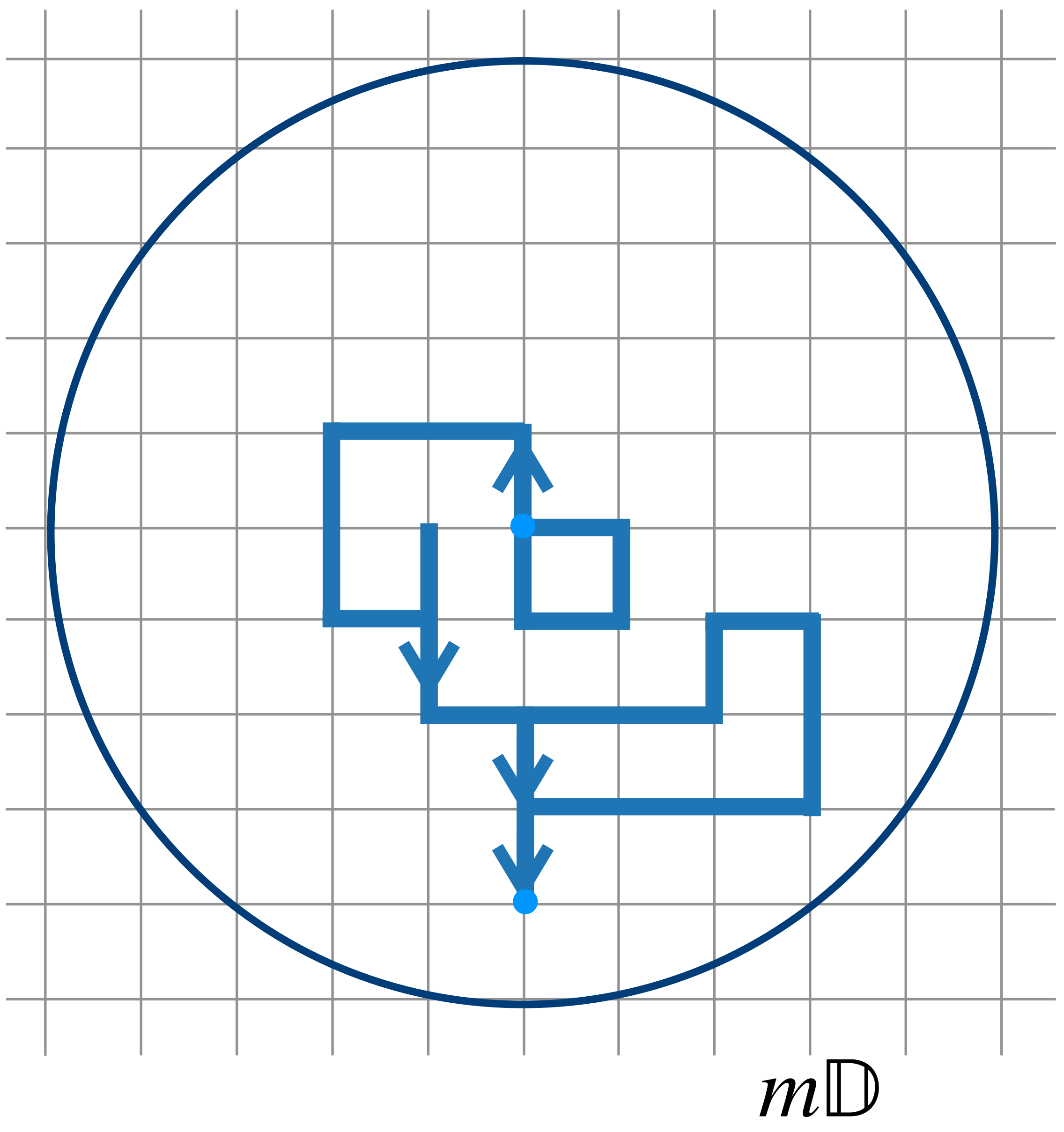

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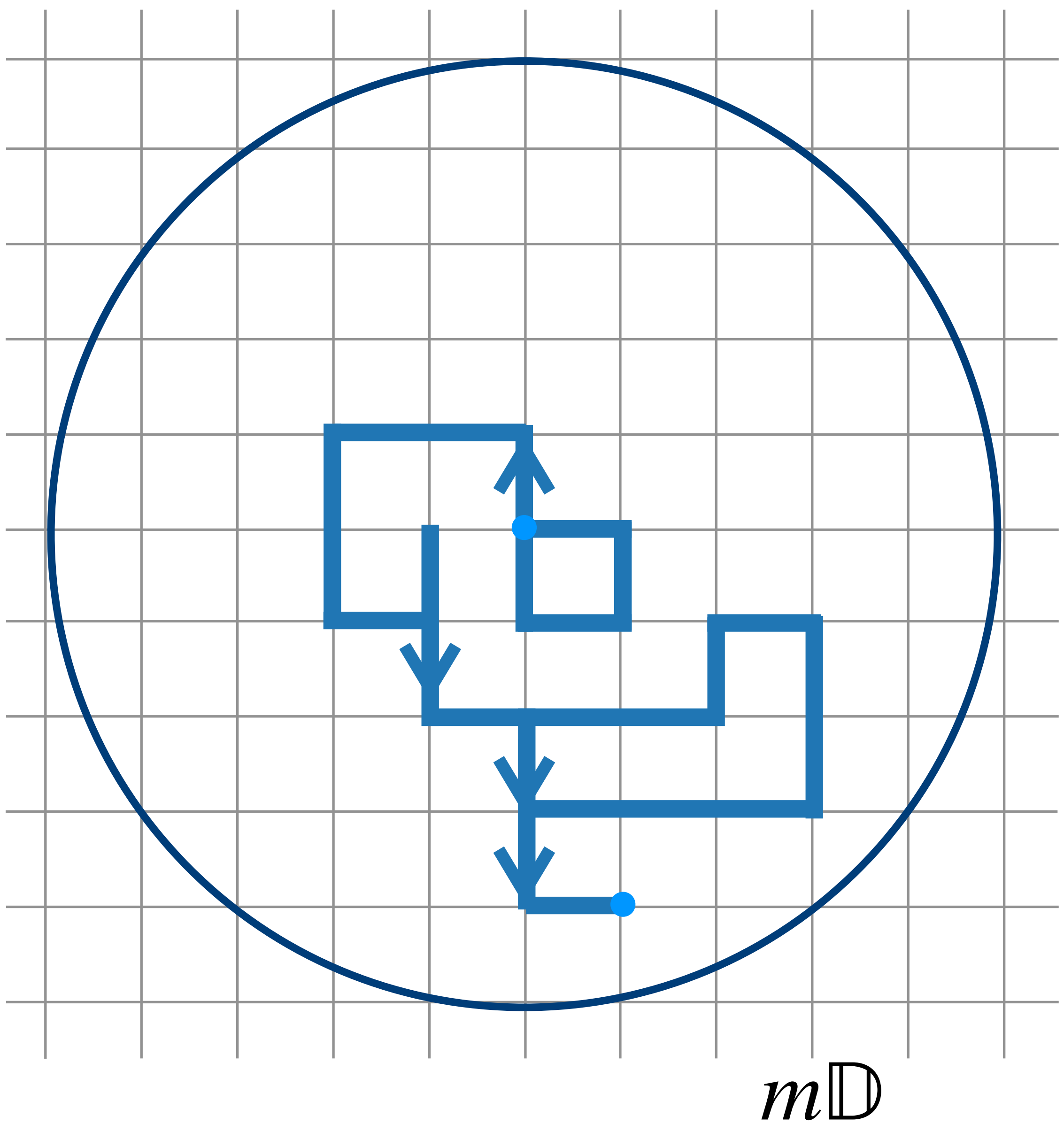

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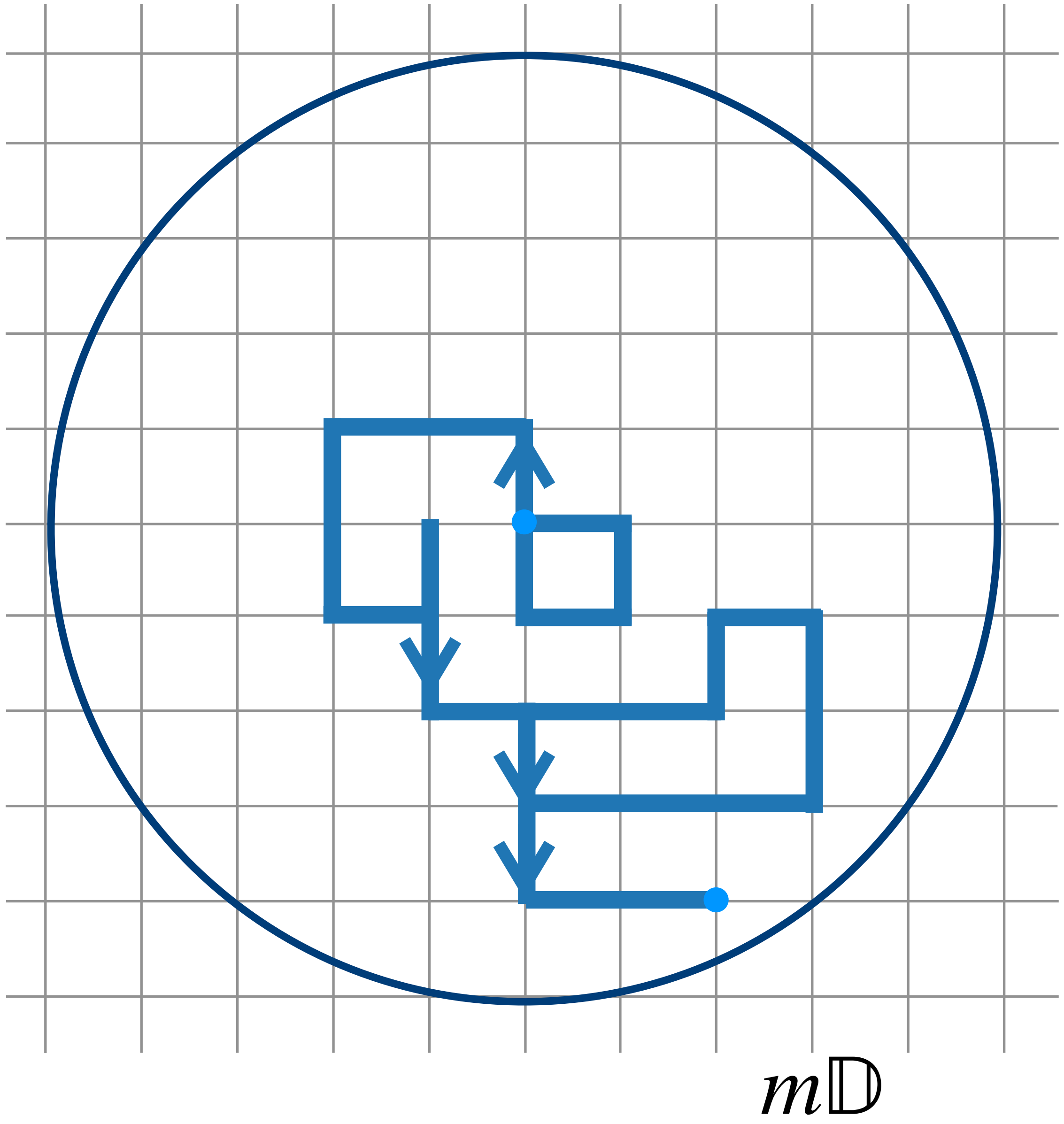

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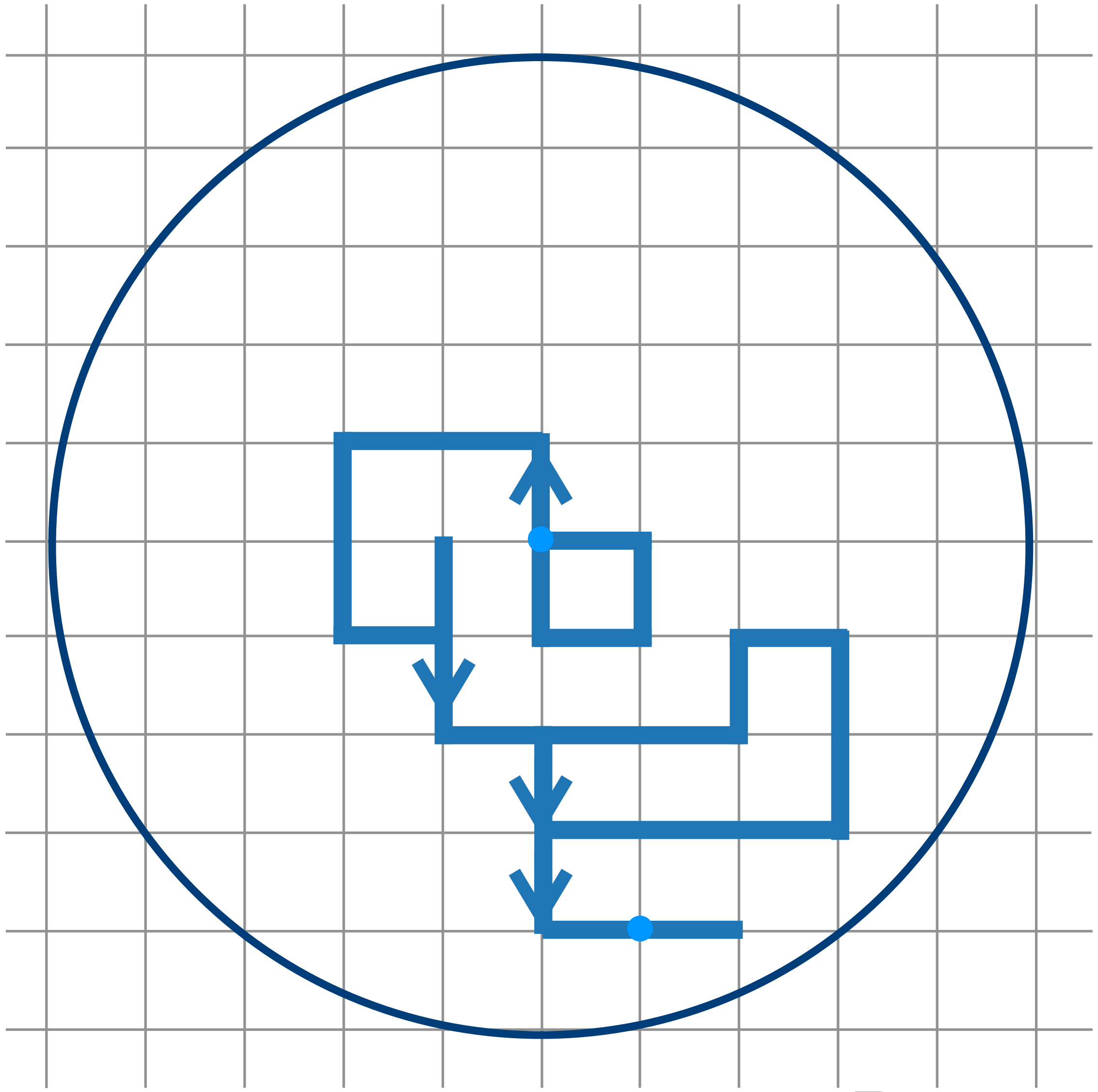

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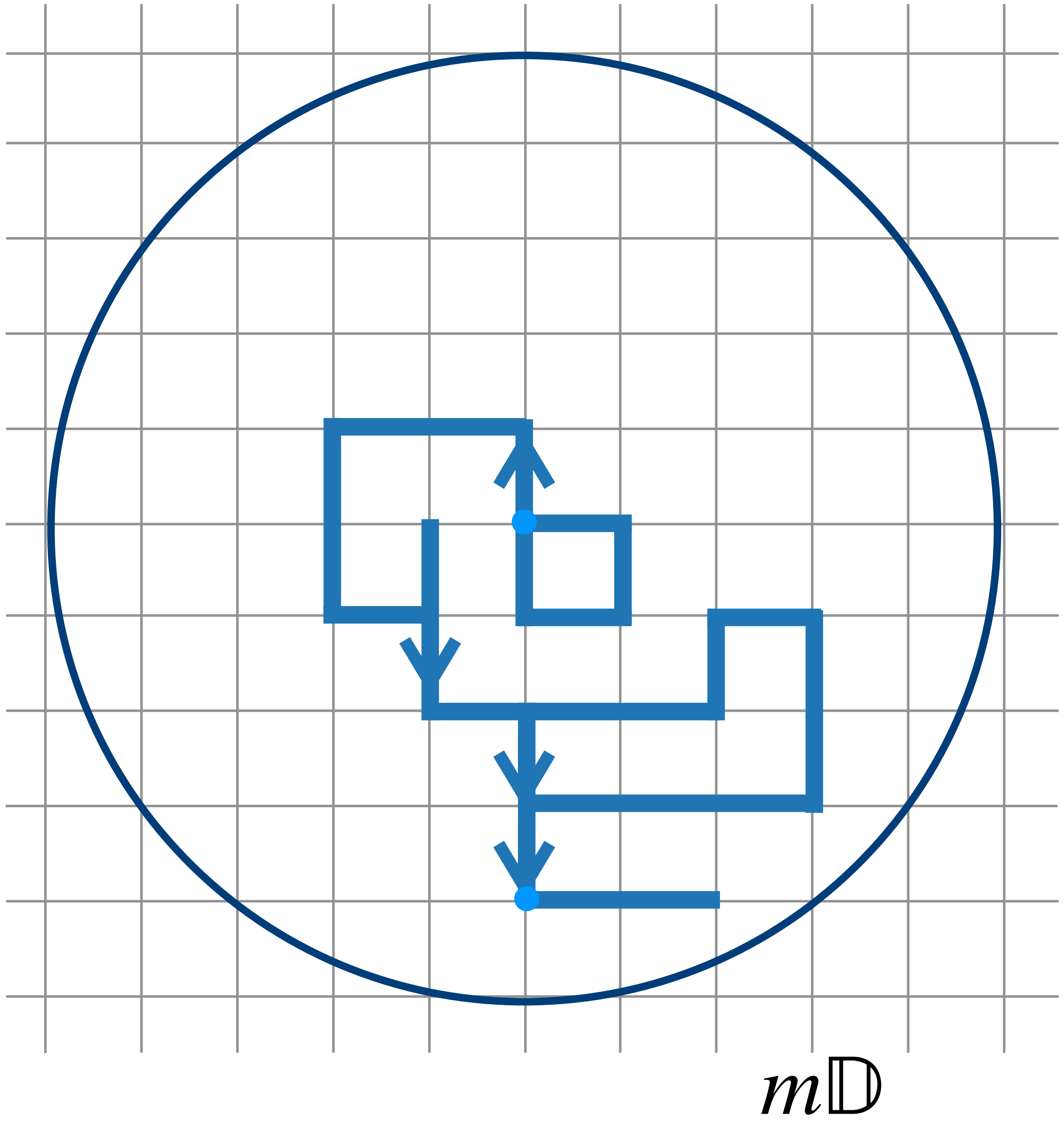

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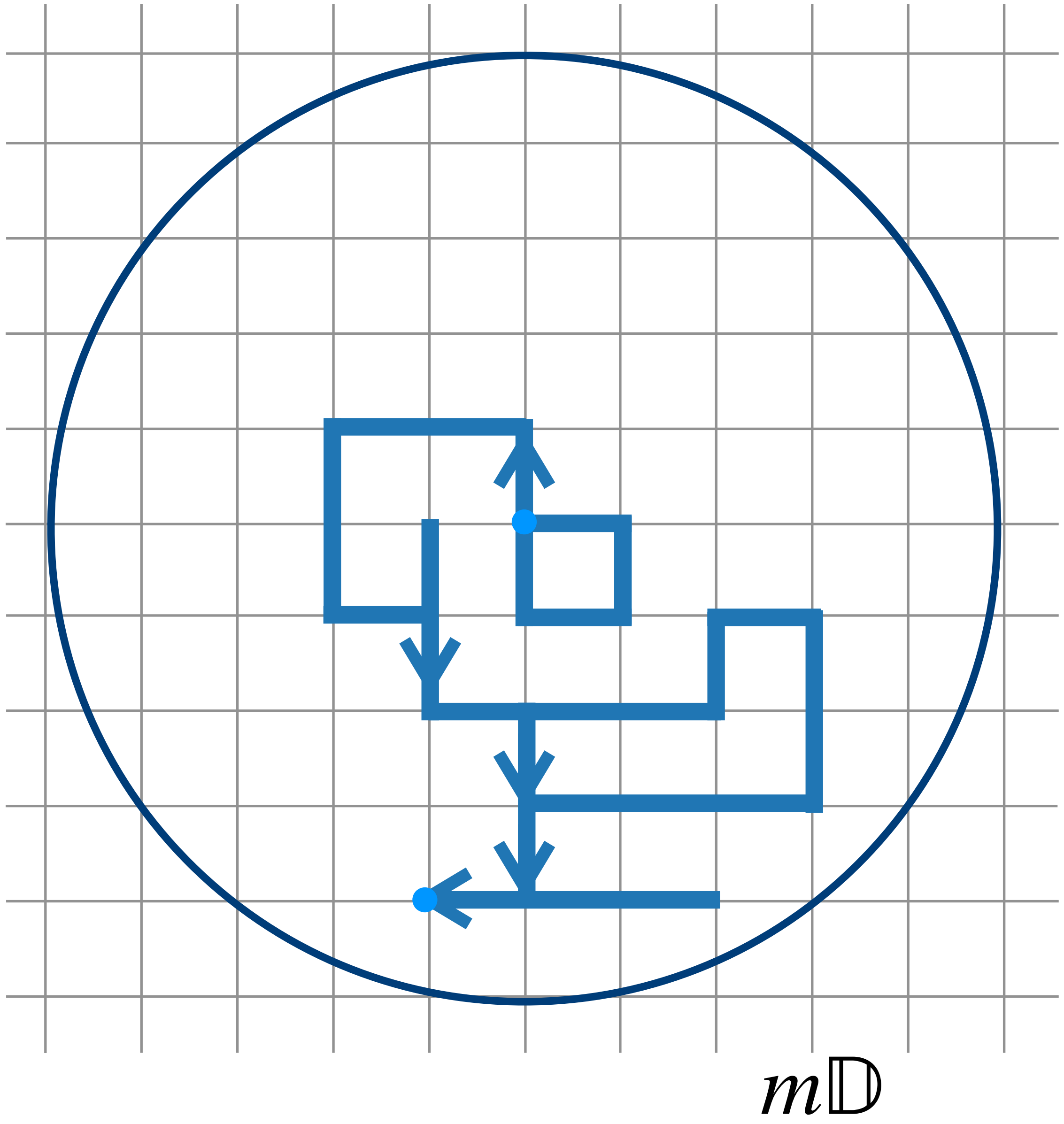

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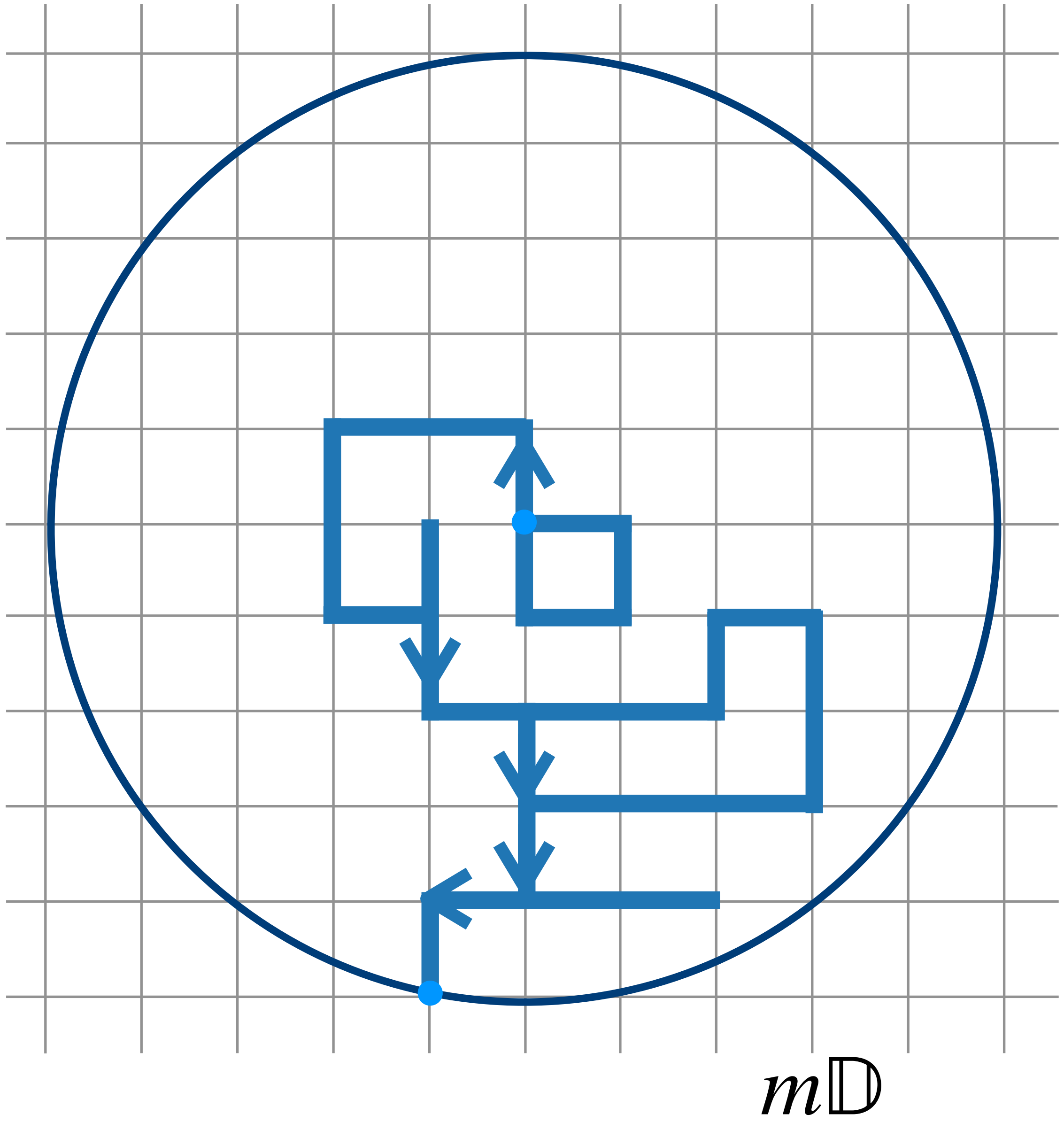

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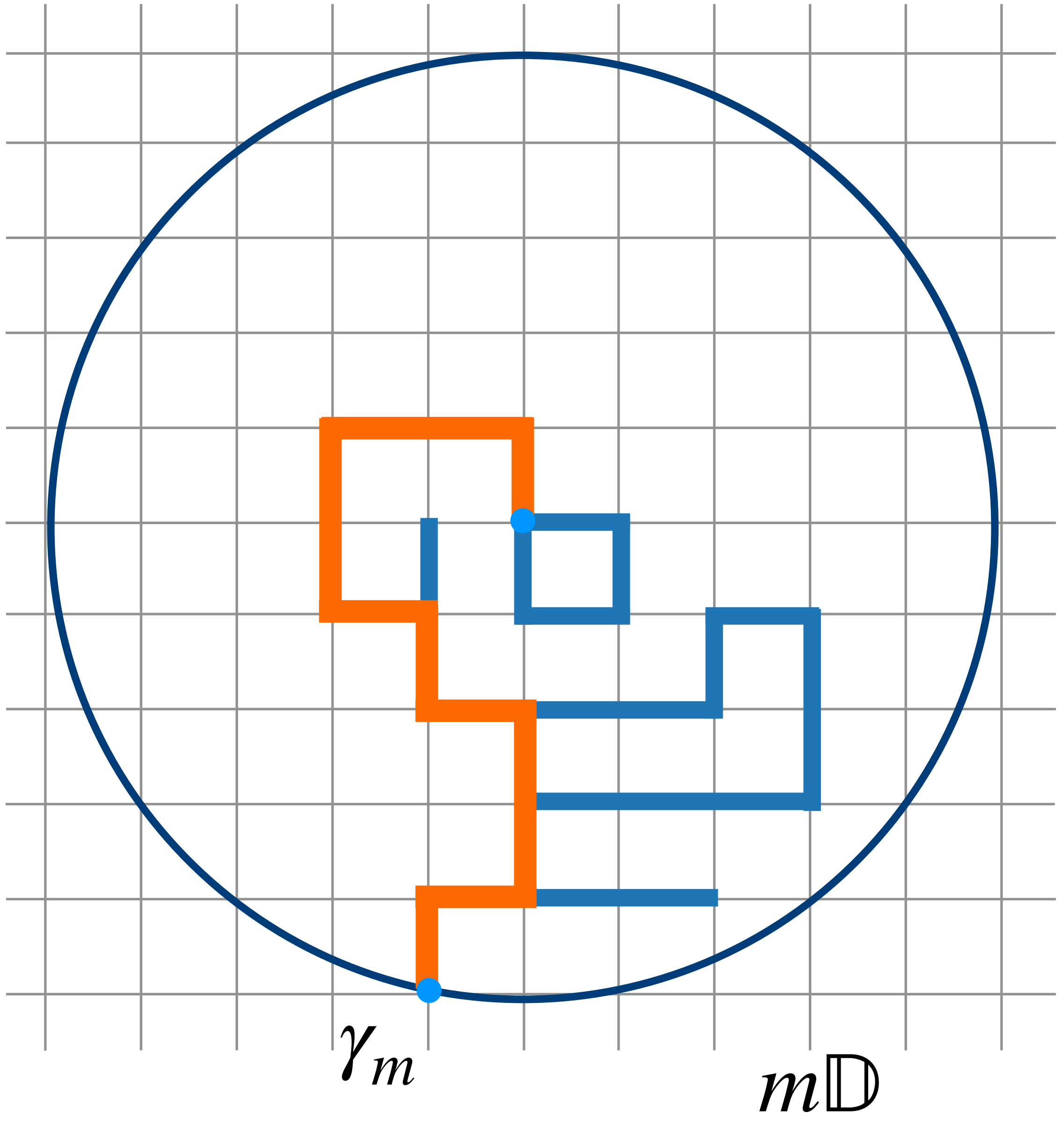

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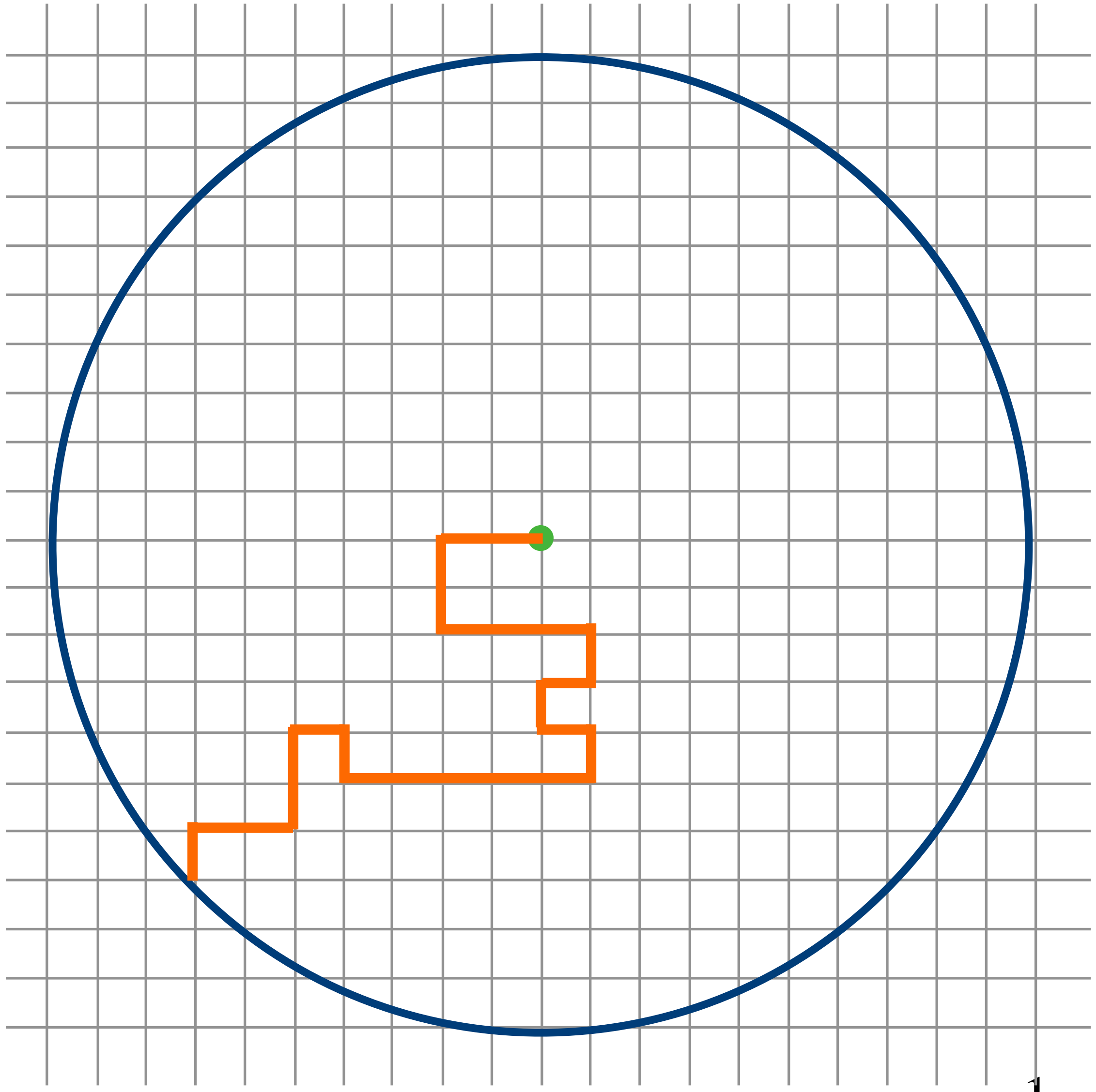

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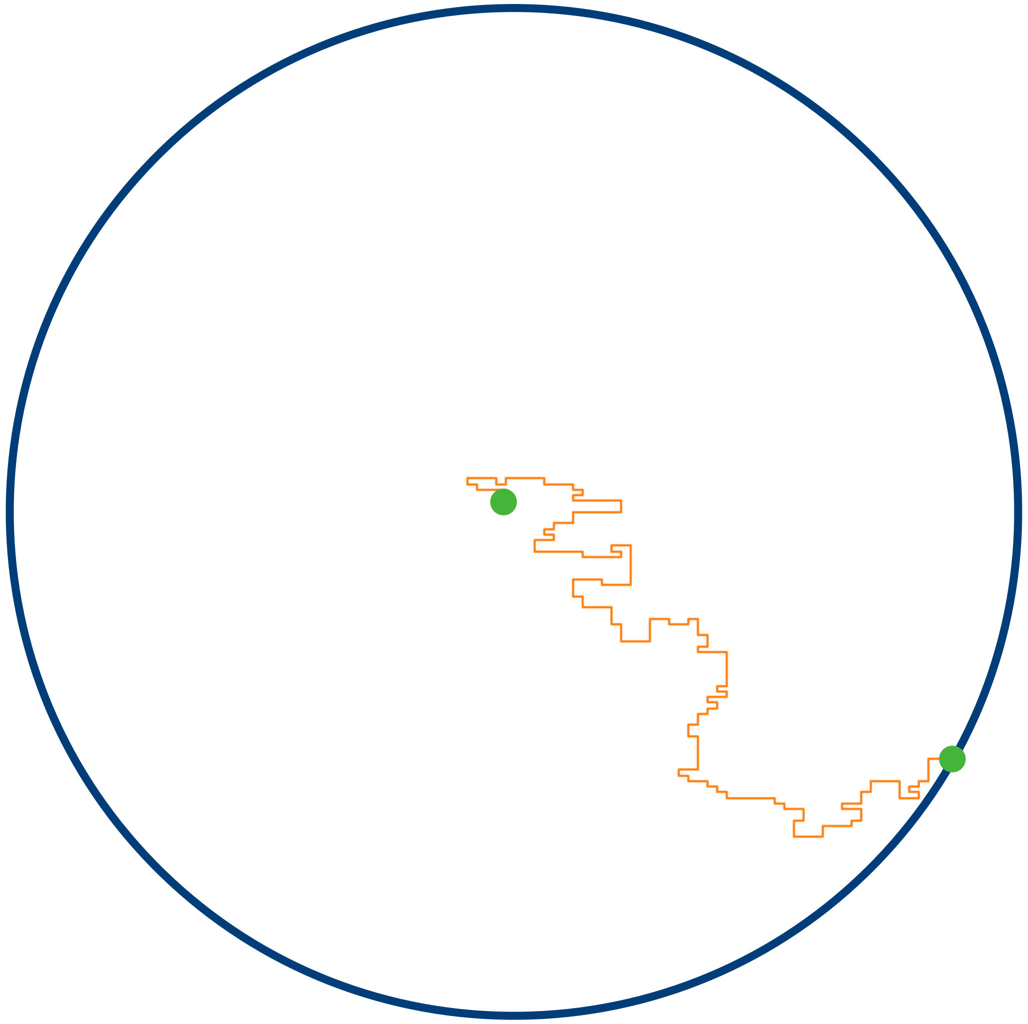

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Simulation: Jorge Emiliano Álvarez Gil Leyva

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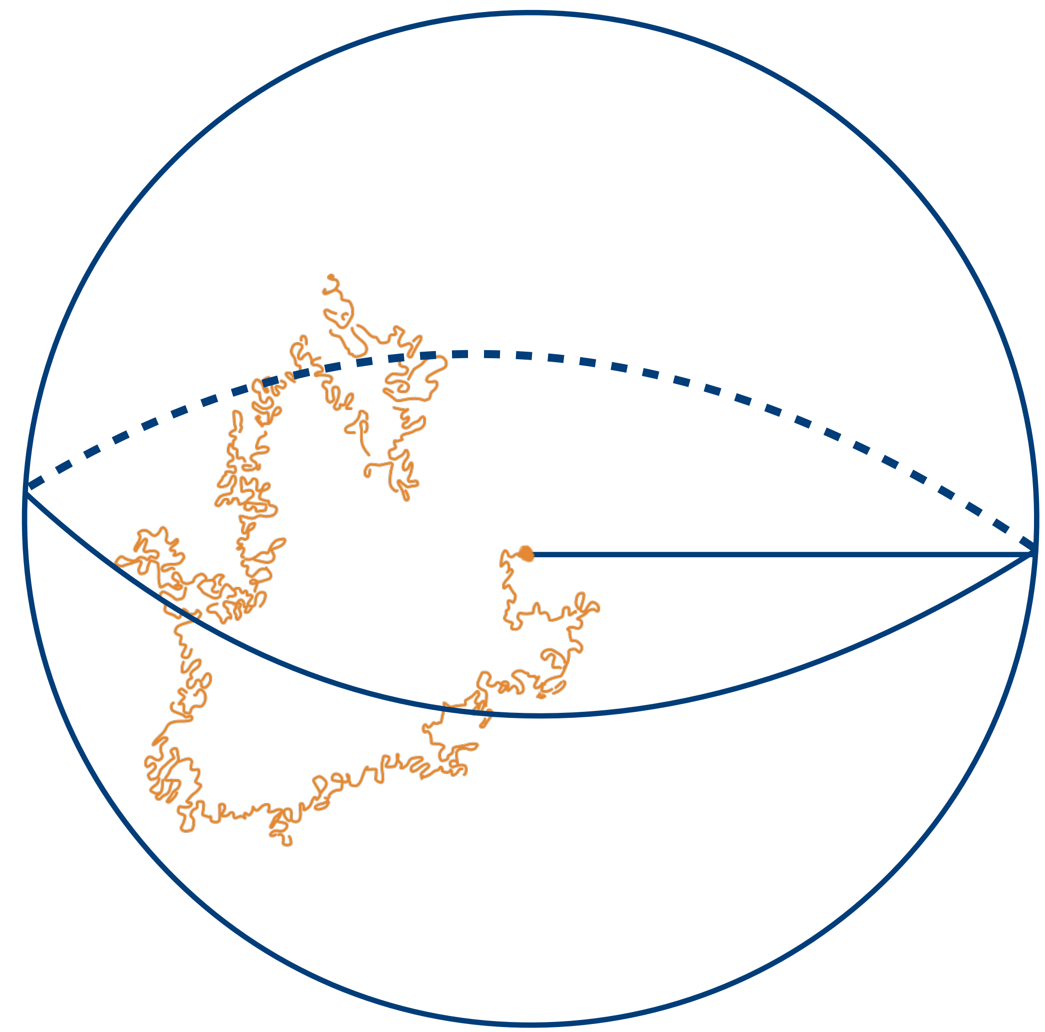

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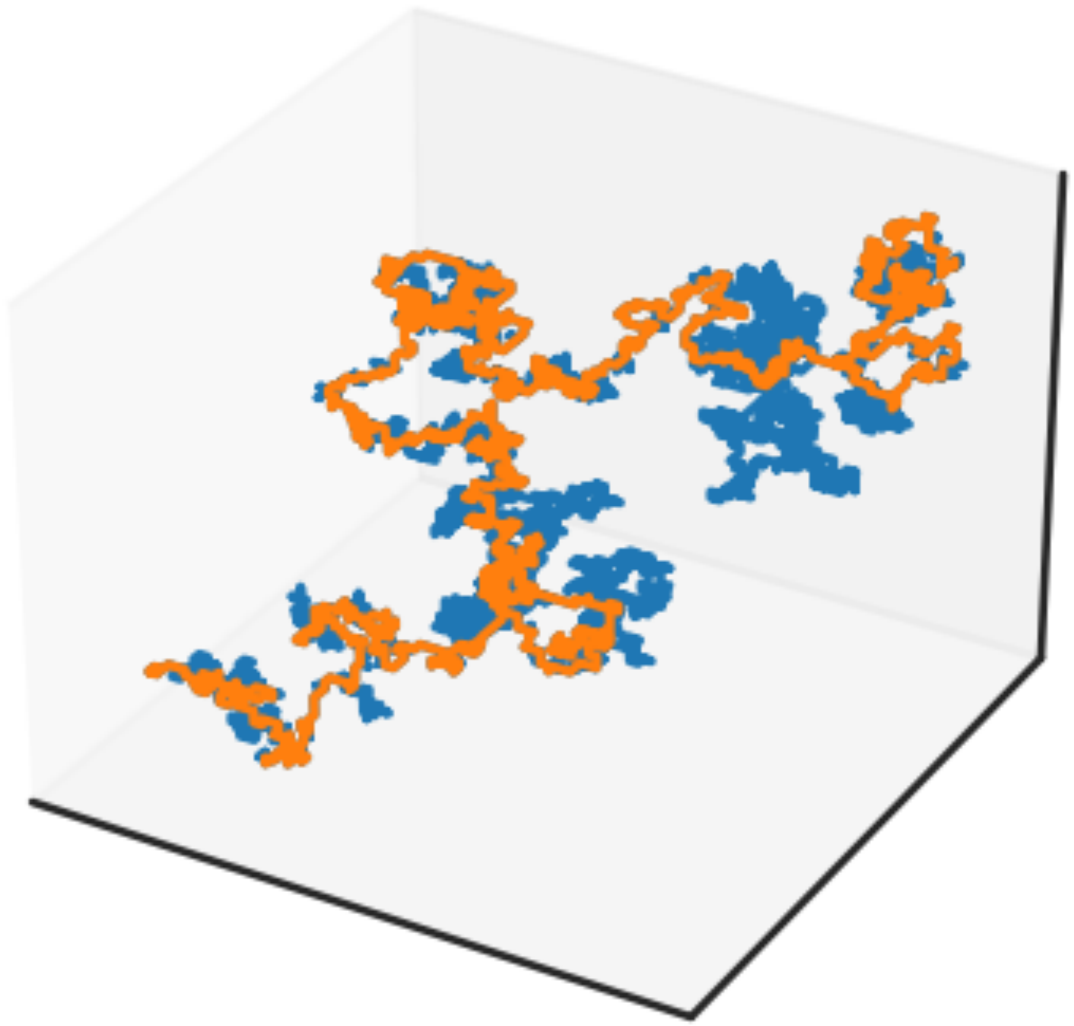

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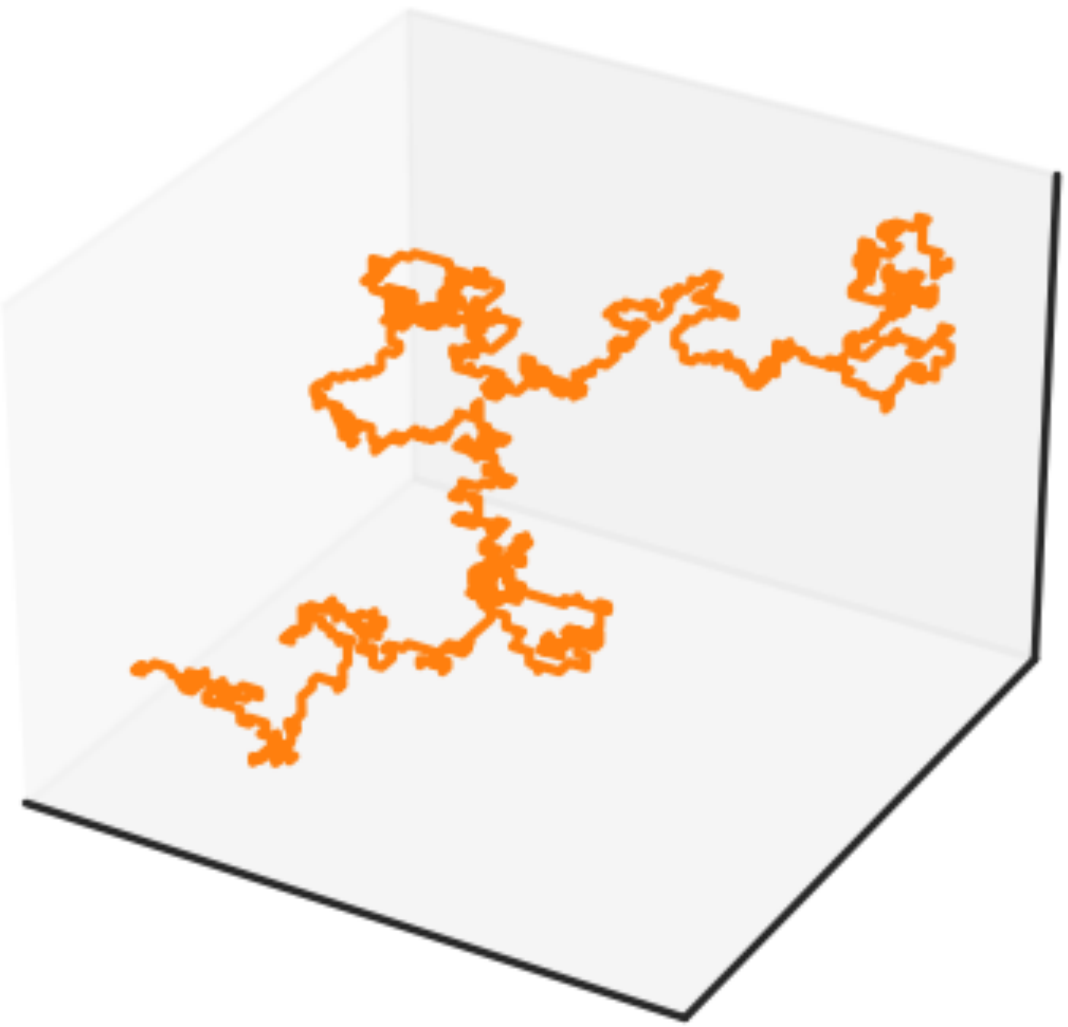

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One-point estimates for LERW

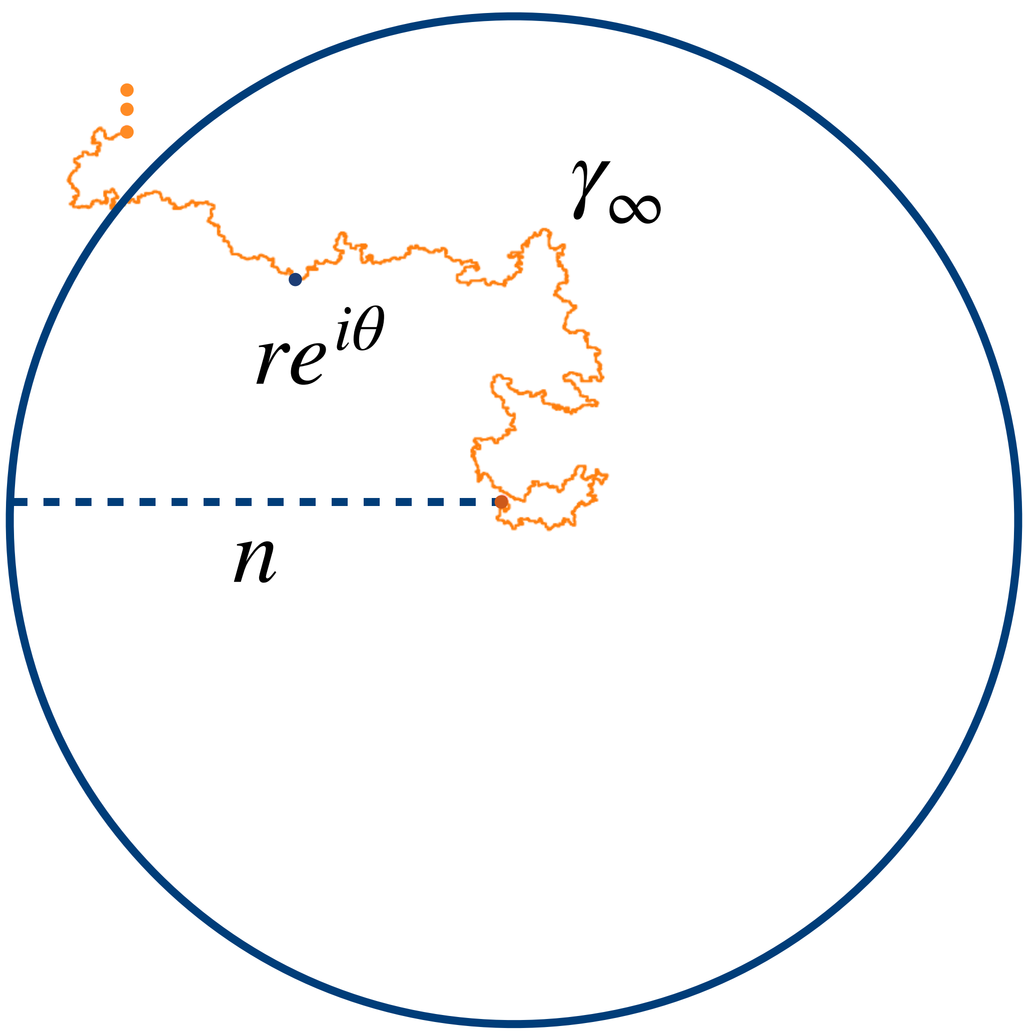
On $\mathbb{D}_m$, we look at $P(x_n \in \gamma_n)$

- Growth exponent
- Scaling limit
- Minkowski content for the scaling limit

LERW in 2D

Growth exponent is $\beta_2 = 5/4$ [Kenyon '00]

- $M_n = \inf\{k \geq 0 : |\gamma_\infty| \geq n\}$
- $\mathbb{E}(M_n) \approx n^{\beta_2} \approx n^{2-3/4}$



$$\mathbb{P}\left(re^{i\theta} \in \gamma_\infty\right) \simeq c_\theta r^{-3/4(1+o_r(1))}$$

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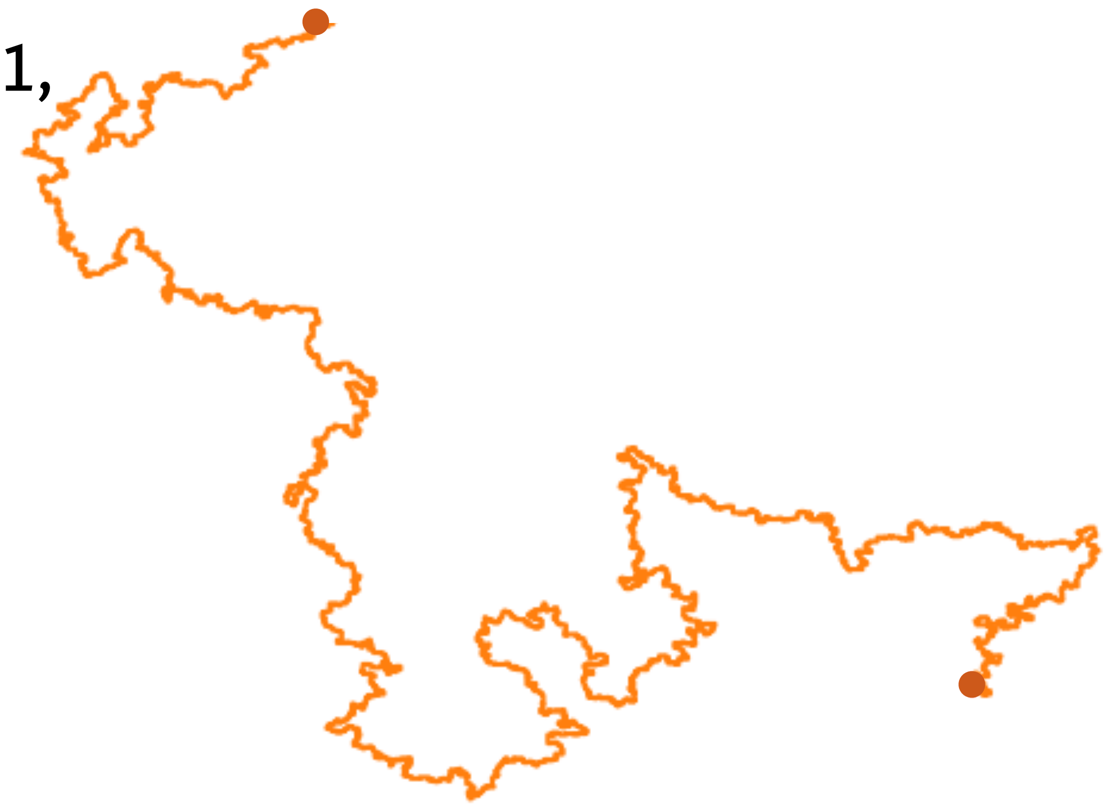
[Schramm '99, Lawler-Schramm-Werner '01, Lawler-Viklund '16]
The scaling limit of LERW on $\mathbb{Z}^2$ is SLE(2)



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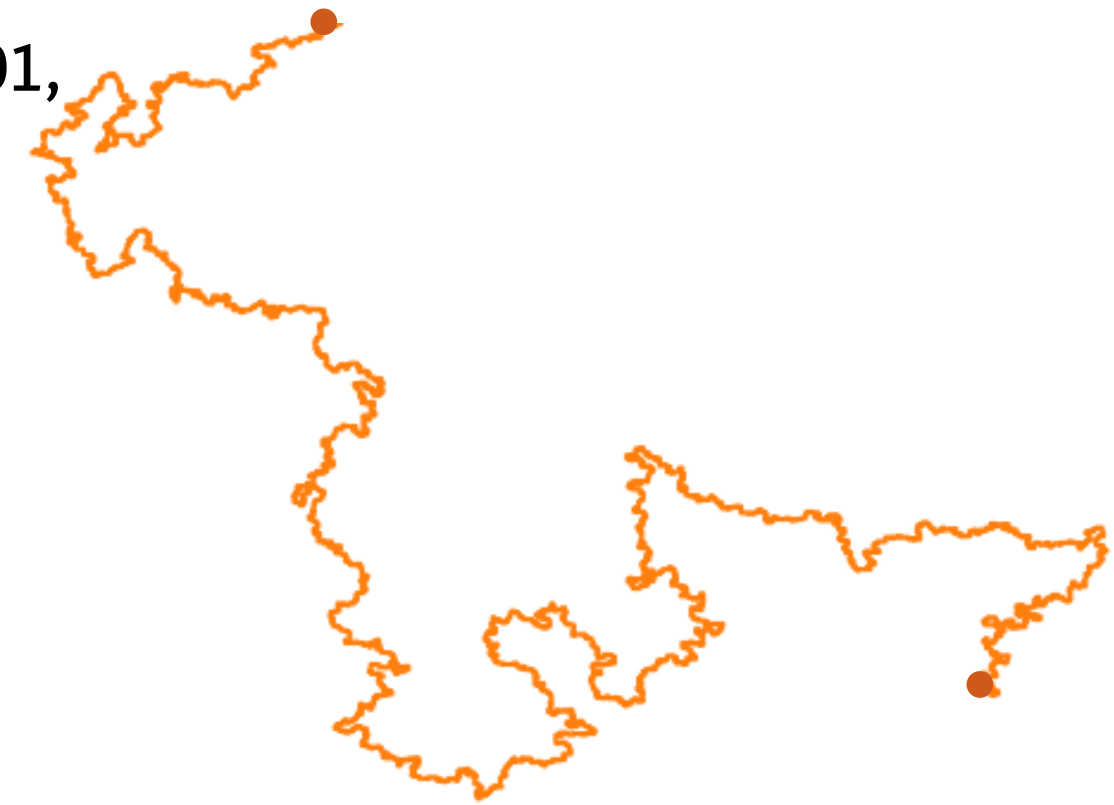
- Convergence in the natural parametrization [Lawler-Viklund '16].
- SLE(2) parametrized by its Minkowski content
[Lawler-Sheffield '09, Lawler-Rezaei '12].



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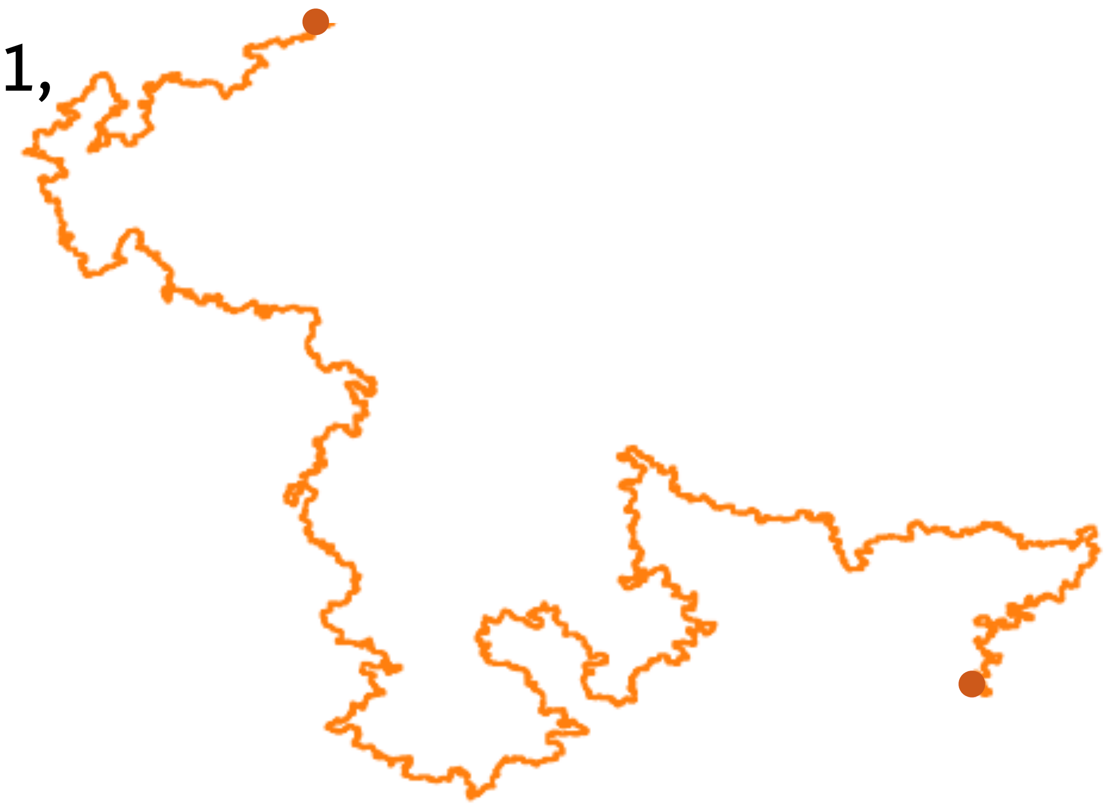
[Schramm '99, Lawler-Schramm-Werner '01, Lawler-Viklund '16]
The scaling limit of LERW on $\mathbb{Z}^2$ is SLE(2)

- Convergence in the natural parametrization [Lawler-Viklund '16].
- SLE(2) parametrized by its Minkowski content [Lawler-Sheffield '09, Lawler-Rezaei '12].
- Strong estimate of one-point function [Benes-Lawler-Viklund '14].



$$\mathbb{P}\left(re^{i\theta} \in \gamma_\infty\right) \sim cr^{-3/4}$$

Simulation: Jorge Emiliano Álvarez Gil Leyva

LERW beyond 2D

Scaling limit		One-point function of LERW
	$d \geq 5$	
	d=4	
	d=3	
SLE(2)	d=2	$P(x_n \in \gamma_n) \sim c n^{-3/4}$ [Benes-Lawler-Viklund ’14]

LERW beyond 2D

Scaling limit		One-point function of LERW
		$x \in \mathbb{D}, x_n \in \mathbb{D}_n$
Brownian motion	d≥5	$P(x_n \in \gamma_n) \sim c n^{2-d}$ [Lawler]
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$\mathscr{K}$	d=3	$P(x_n \in \gamma_n) \sim cn^{\beta-3}$
SLE(2)	d=2	$P(x_n \in \gamma_n) \sim cn^{-3/4}$ [Benes-Lawler-Viklund '14]

Scaling limit of the 3D LERW

[Kozma '05] Convergence of trace

- Let $P \subset \mathbb{R}^3$ be a polyhedron.
- $P_m = P \cap m^{-1}\mathbb{Z}^3$
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$\gamma_{2^{-n}} \Rightarrow \mathscr{K}$ w.r.t. the Hausdorff metric.

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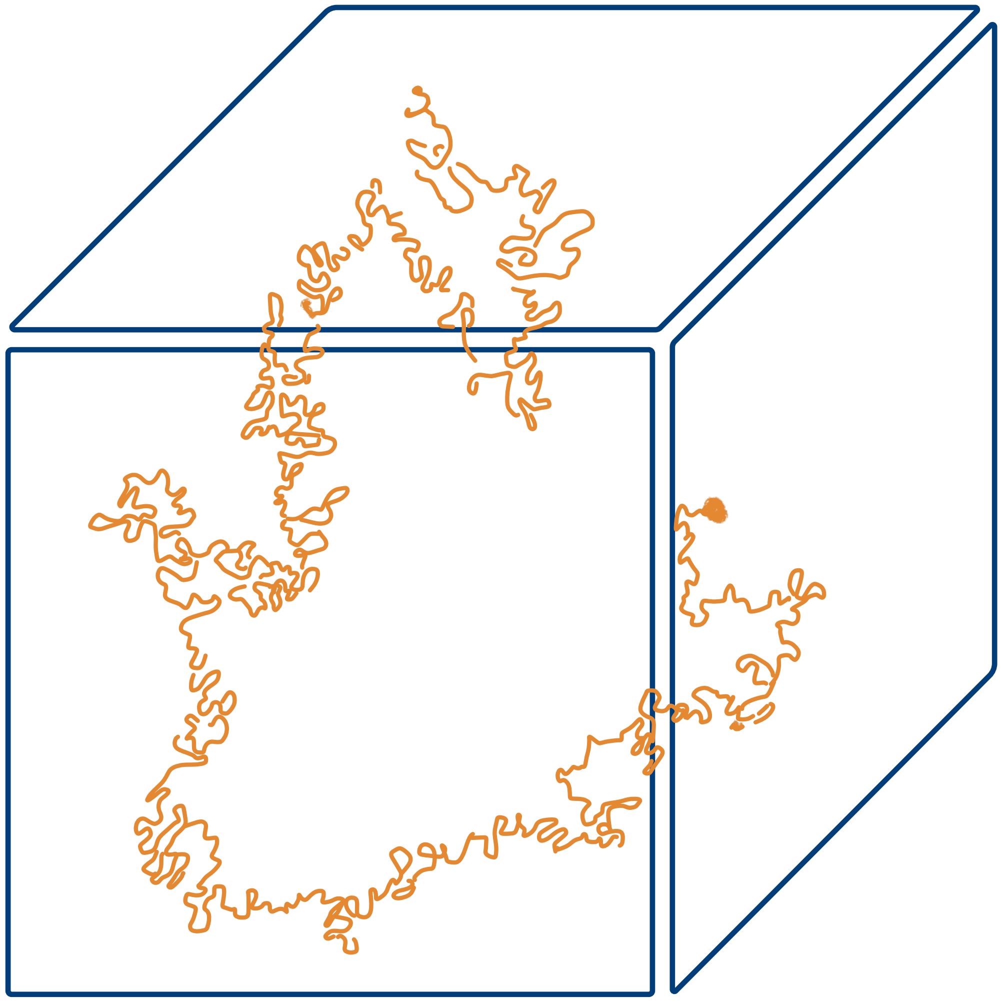

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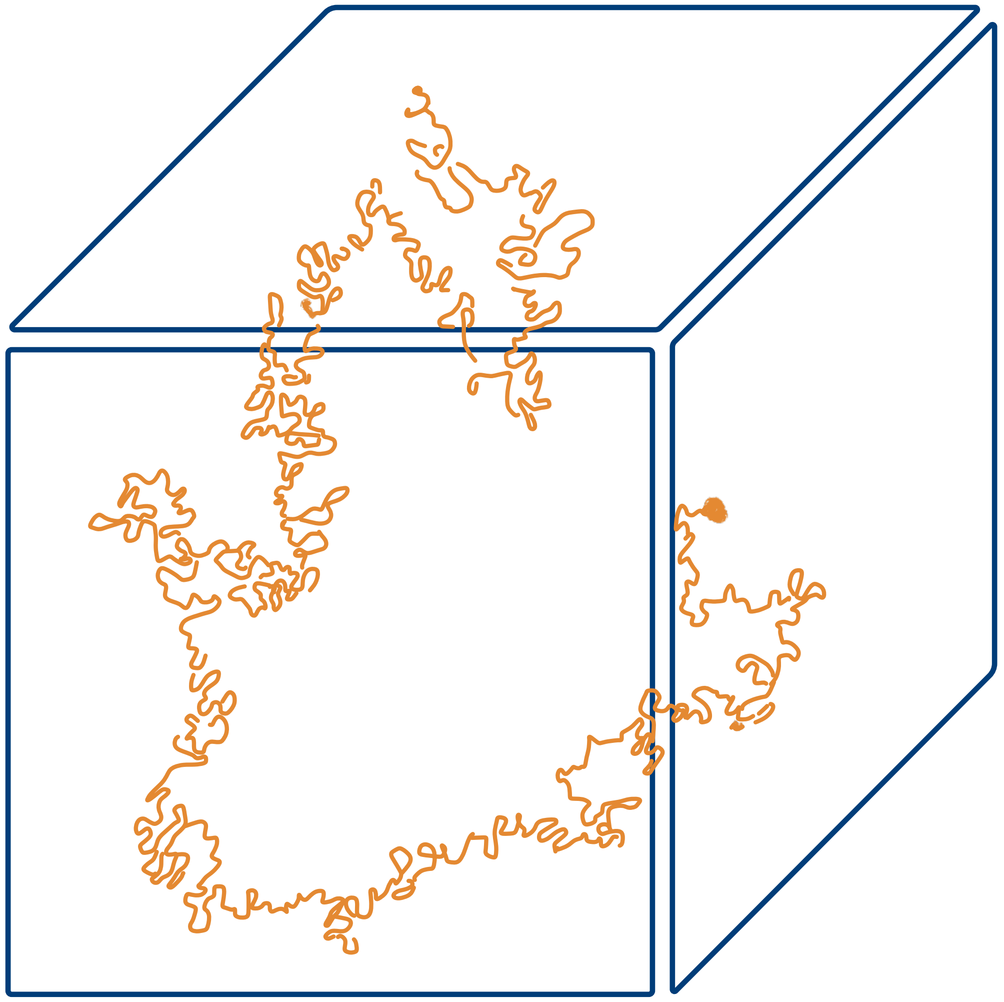


[Sapozhnikov-Shiraishi '15]

$\mathscr{K}$ is a simple curve a.s.

Growth exponent of 3D LERW

[Shiraishi '13]

- Let $M_n = \inf\{k \geq 0 : |\gamma_\infty| \geq n\}$

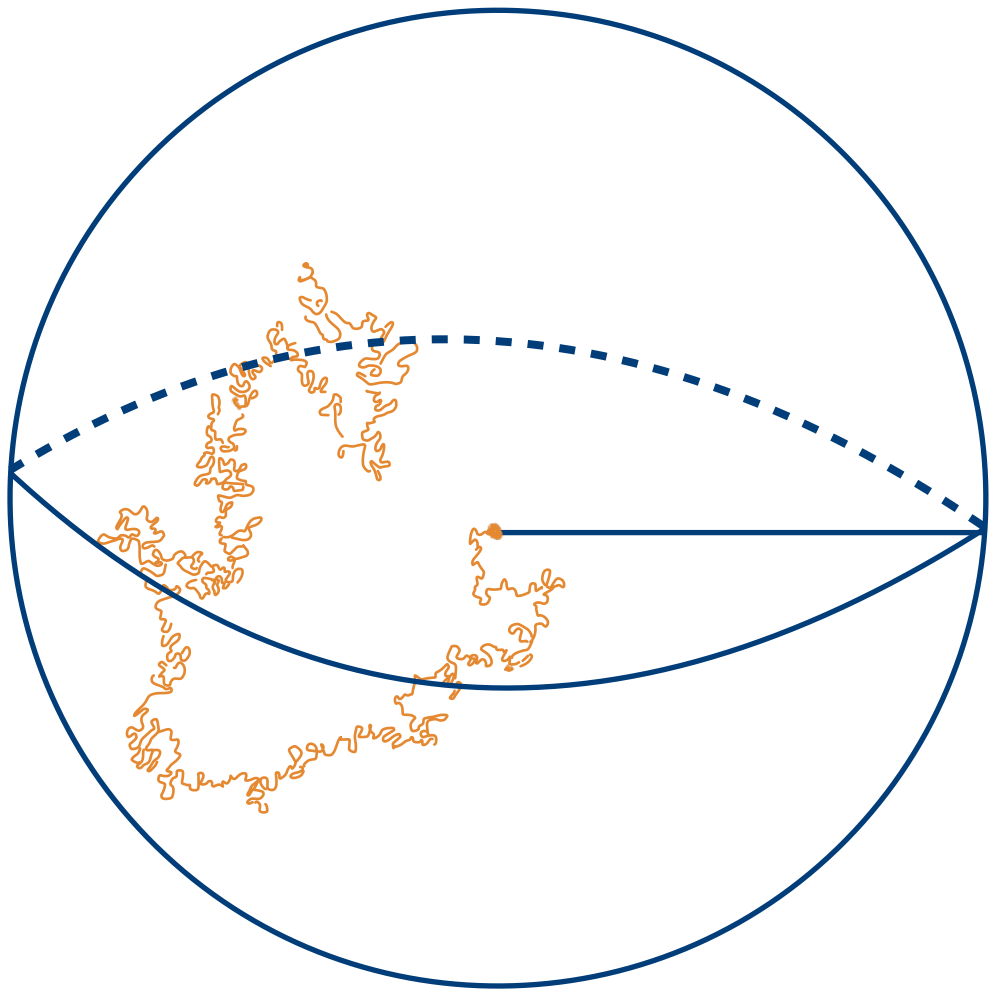
Then there exists $\beta \in (1, 5/3]$ such that

$$\mathbb{E}(M_n) = n^{\beta + o(1)}$$

[Shiraishi '16] $\dim_H \mathscr{K} = \beta$

[Shiraishi '13]

$$P(x_n \in \gamma_n) = n^{\beta - 3 + o(1)}$$



Scaling limit of the 3D LERW

[Li-Shiraishi '18]

- β is the growth exponent of the 3D LERW
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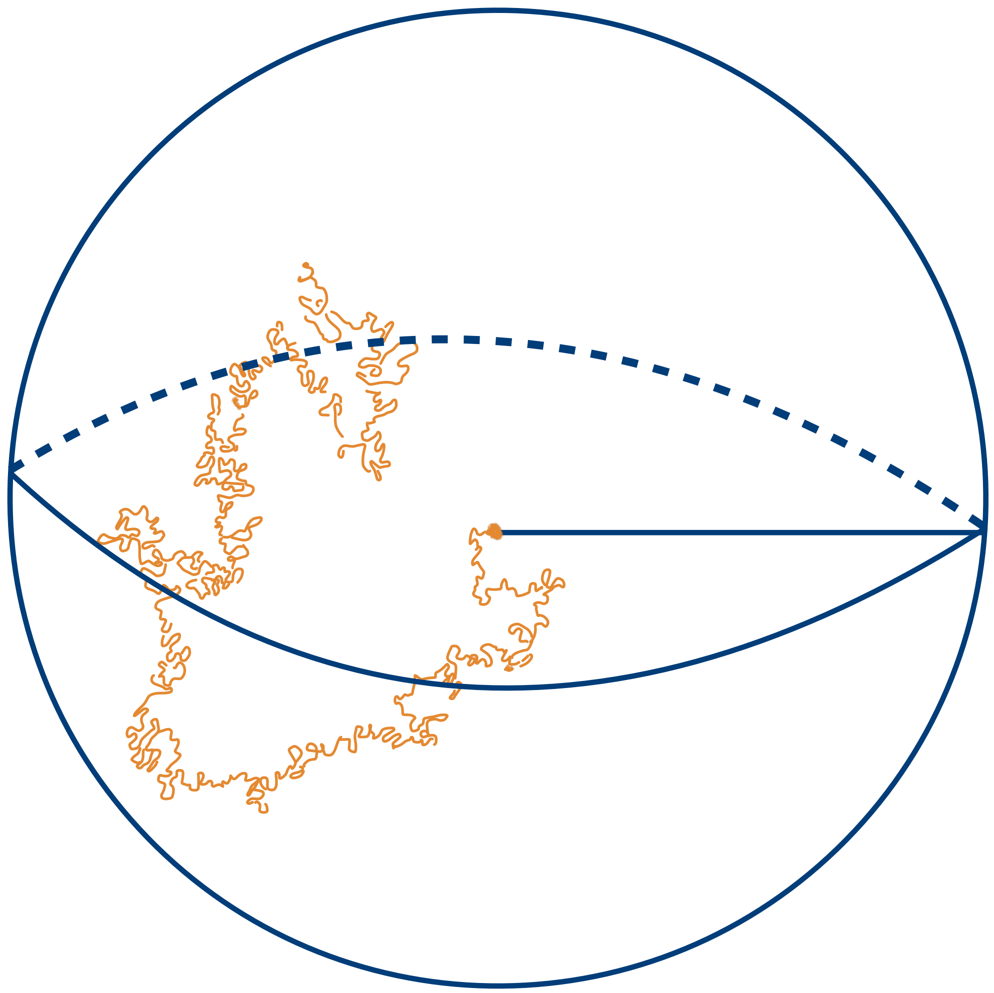

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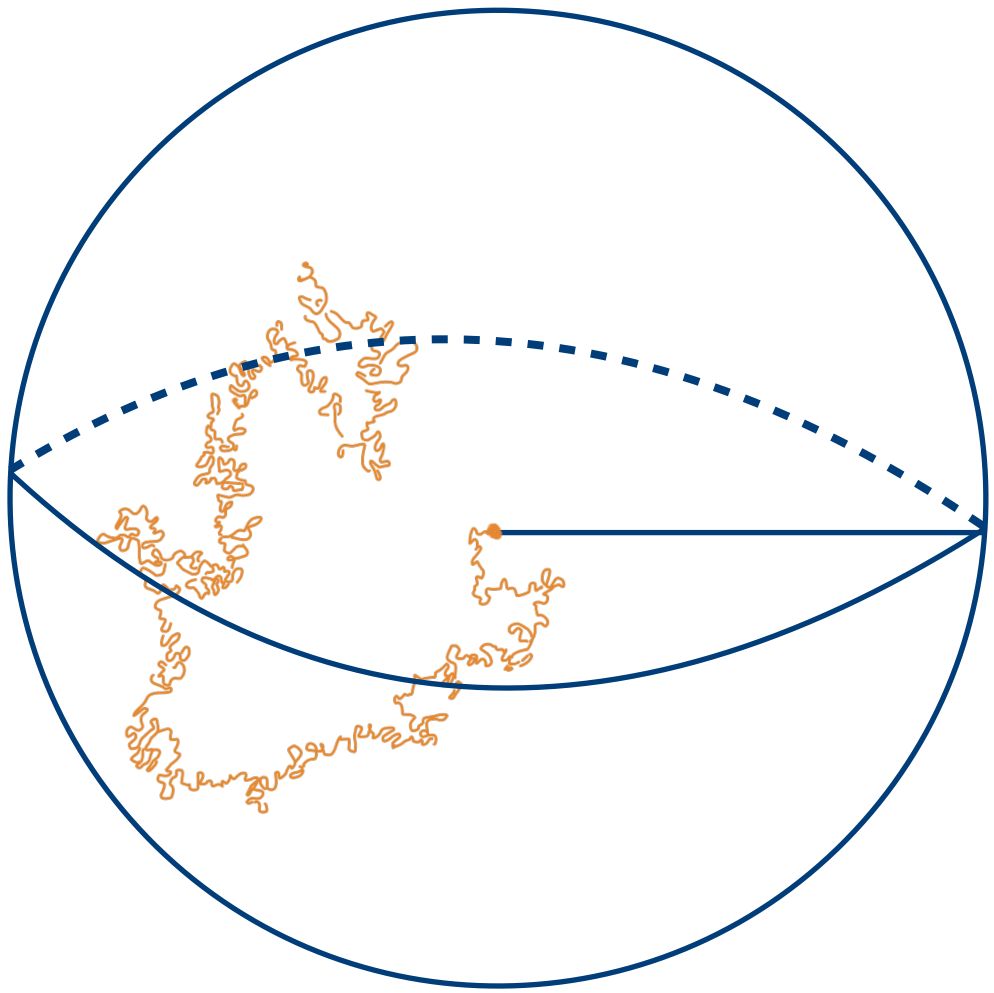

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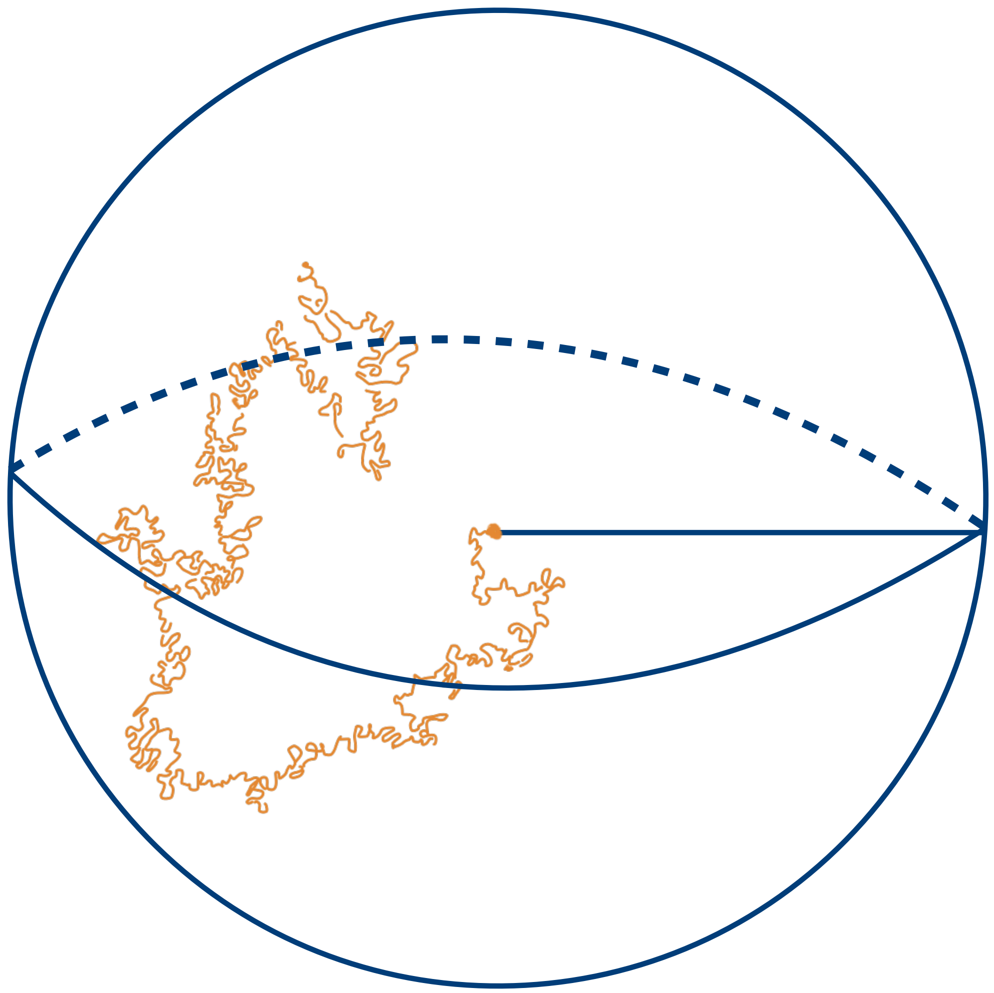

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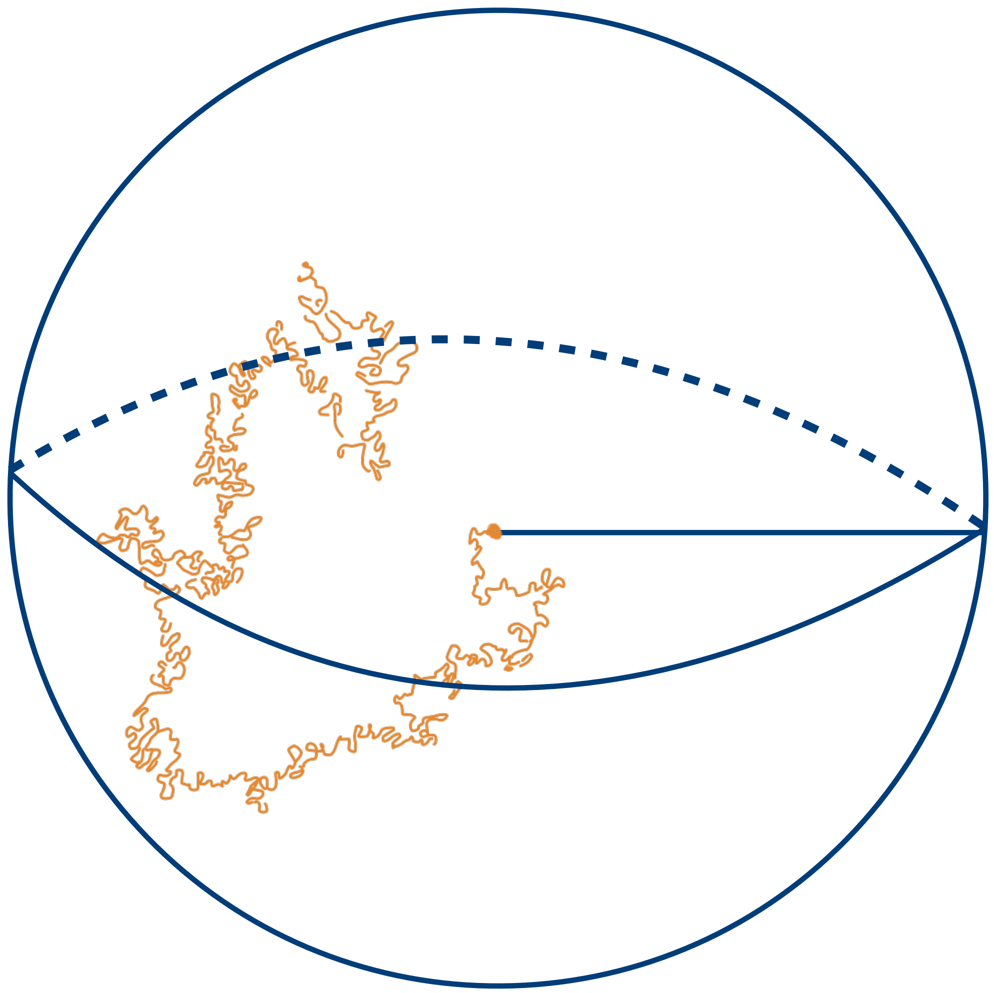

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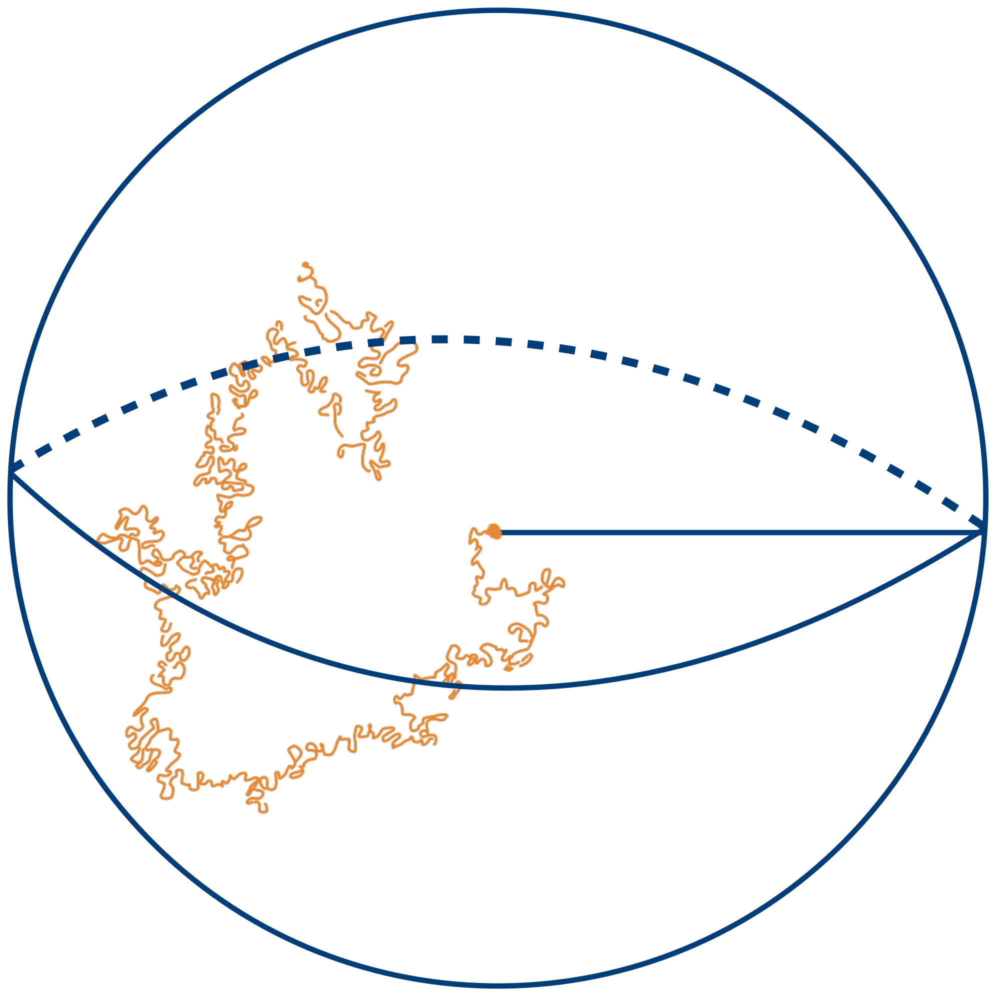

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In the topology generated by the metric

$$\rho(\lambda_1, \lambda_2) = |T_1 - T_2| + \sup_{0 \le s \le 1} |\lambda_1(sT_1) - \lambda_2(sT_2)|,$$

$$\lambda_i : [0, T_i] \to \mathbb{R}^3$$

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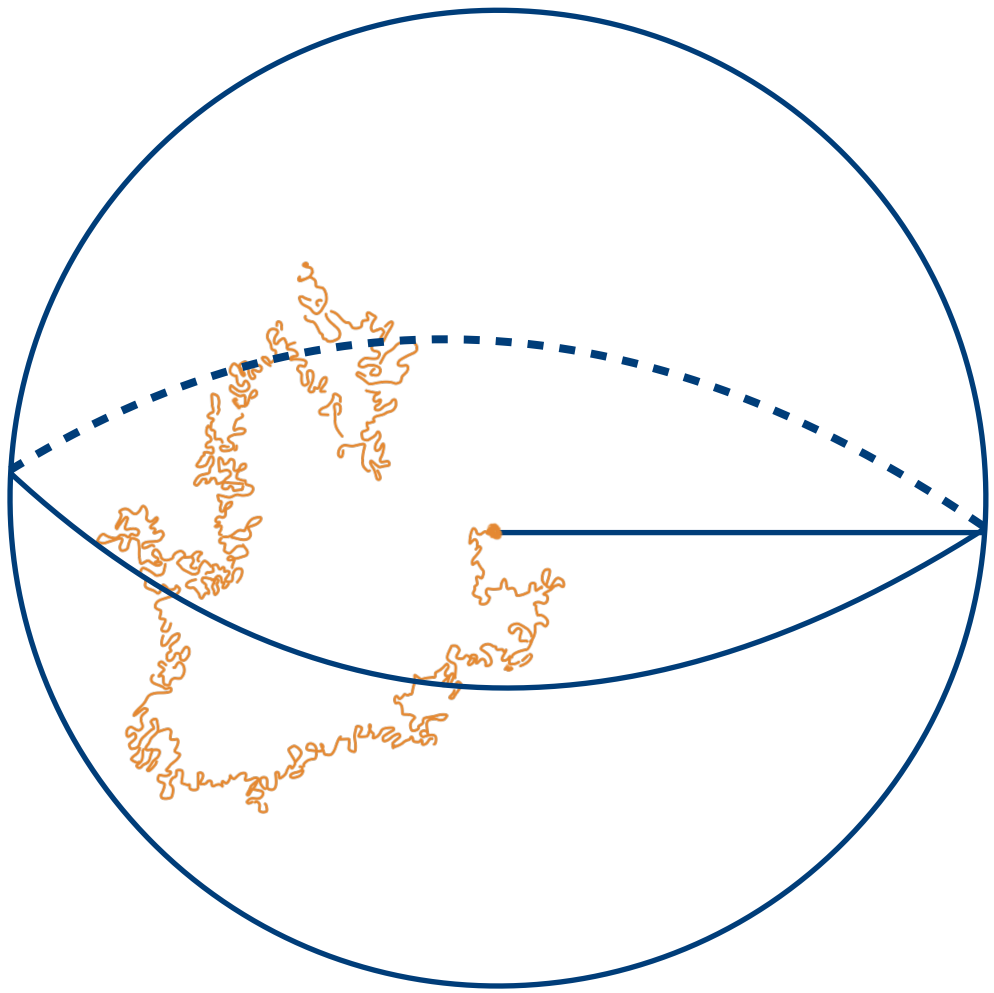

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One-point estimate on dyadic sequences

$P\left(x_{2^n} \in \gamma_{2^n}\right) \simeq c_x 2^{-(3-\beta)n}$ [Li-Shiraishi '18]

One-point function of the 3D LERW

[H.T.-Li-Shiraishi, '24]

- Ball-hitting probability for $\mathscr{K}$

$$P(\mathscr{K} \cap B(x, r) \neq \emptyset) = c g(x) r^{3-\beta} \left[1 + O\left(d_x^{-c} r^{\delta} \right) \right]$$

- One-point estimate

$$P(x_m \in \mathscr{K}_m) = c_1 g(x) m^{-(3-\beta)} \left[1 + O\left(d_x^{-c} m^{-\delta} \right) \right]$$

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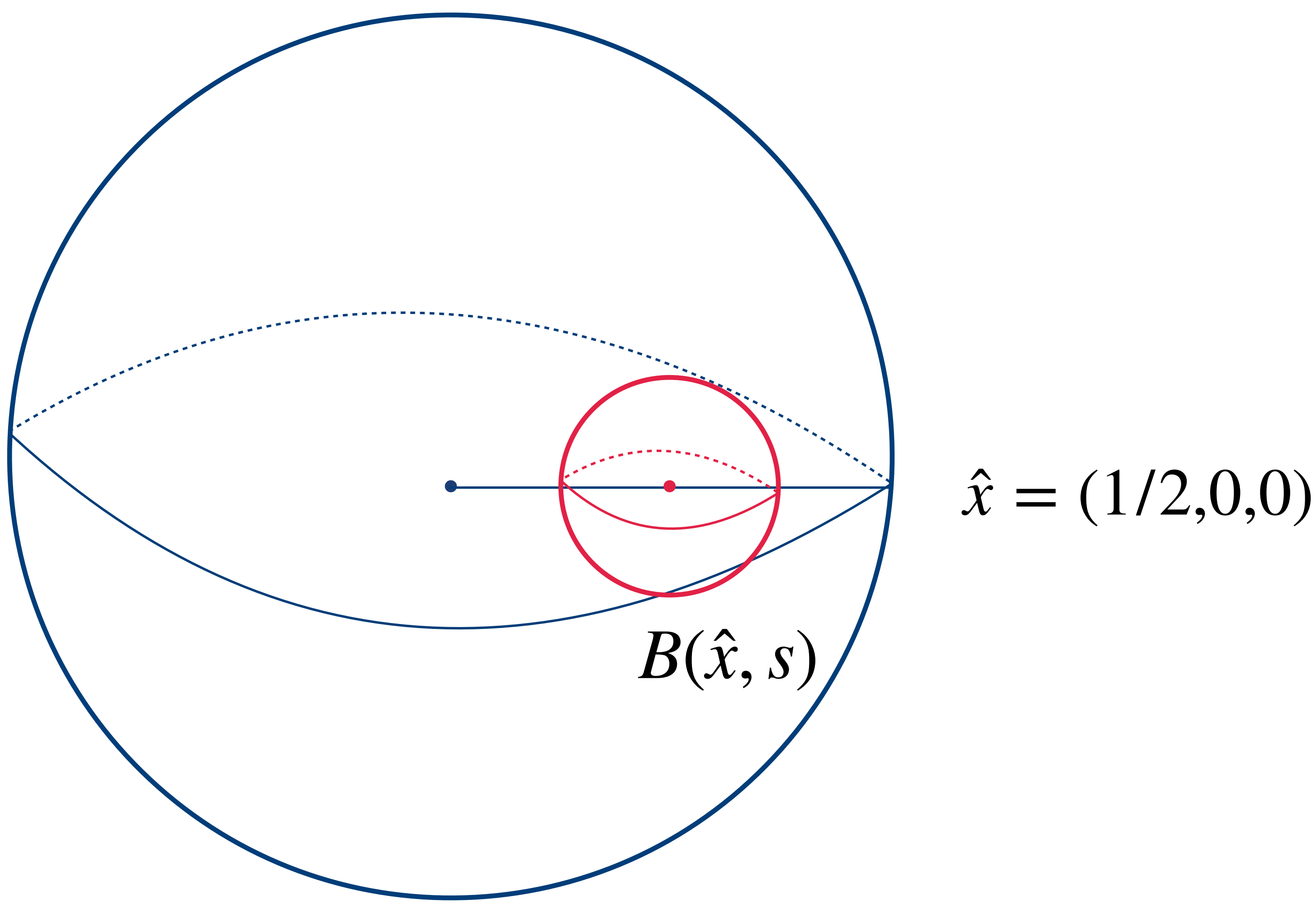
- Continuity result

$$P\left(\mathscr{K} \cap B(x,r) \neq \emptyset, \mathscr{K} \cap B^{\circ}(x,r) = \emptyset \right) = 0$$

Proposition. For each $s > 0$ there exists $m_0 > 0$ so that for all $m > m_0$

$$P(\mathscr{K}_m \cap B(\hat{x}, s) \neq \varnothing) = c_3 s^{3-\beta} \left[1 + O(s^\delta)\right]$$



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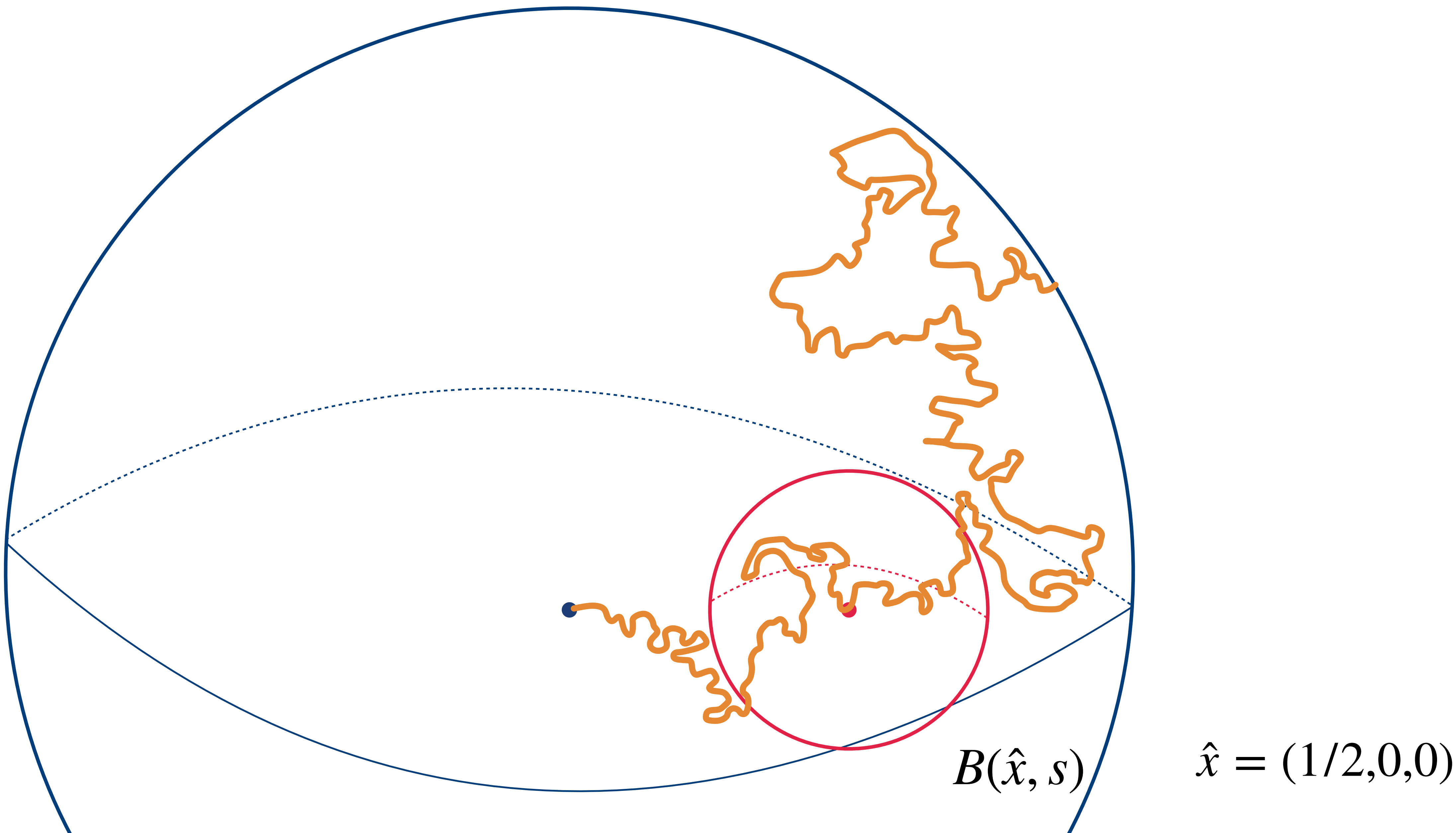

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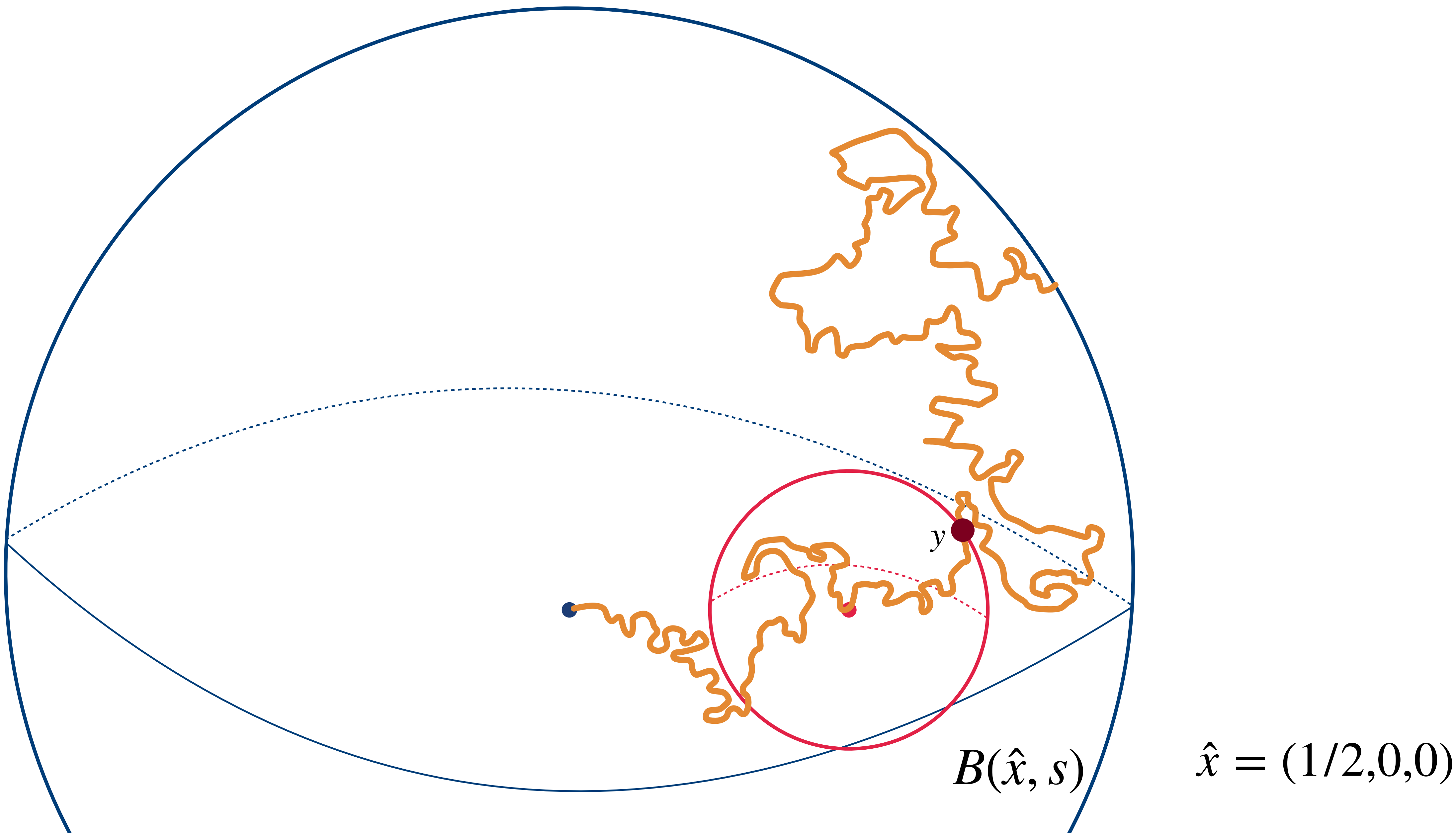

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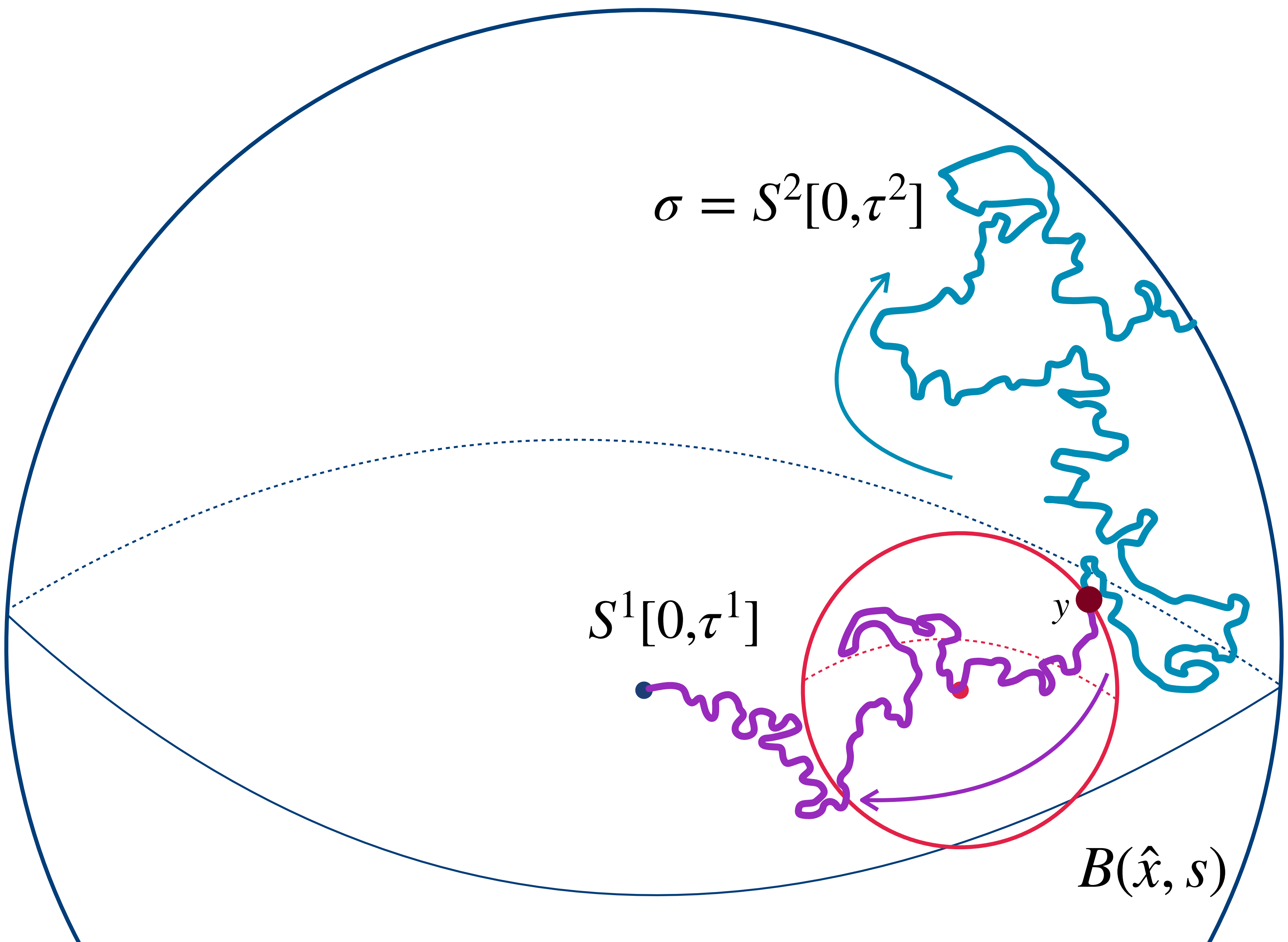

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$S^1[0,\tau^1]$ hits 0

$\hat{x} = (1/2,0,0)$

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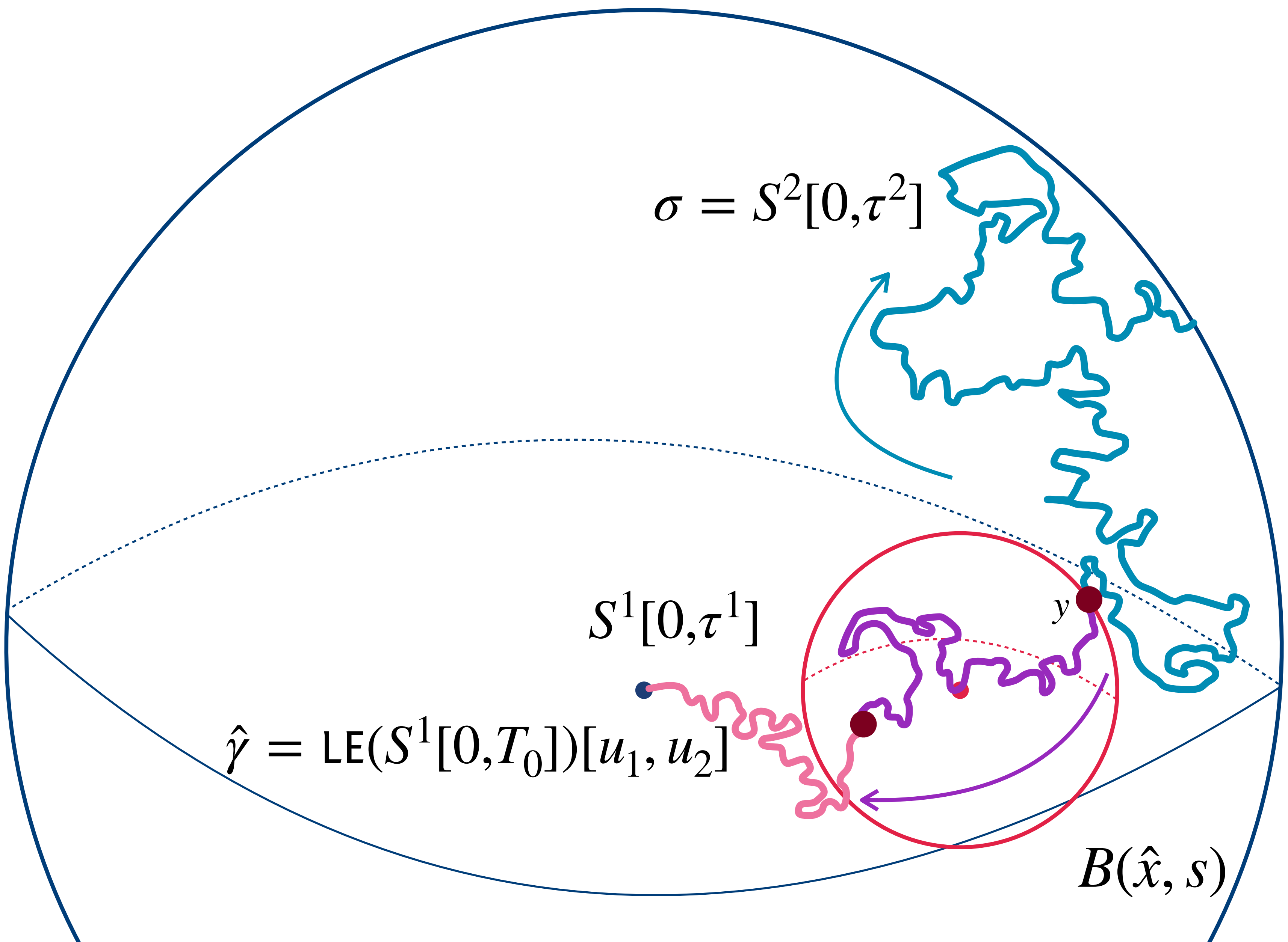

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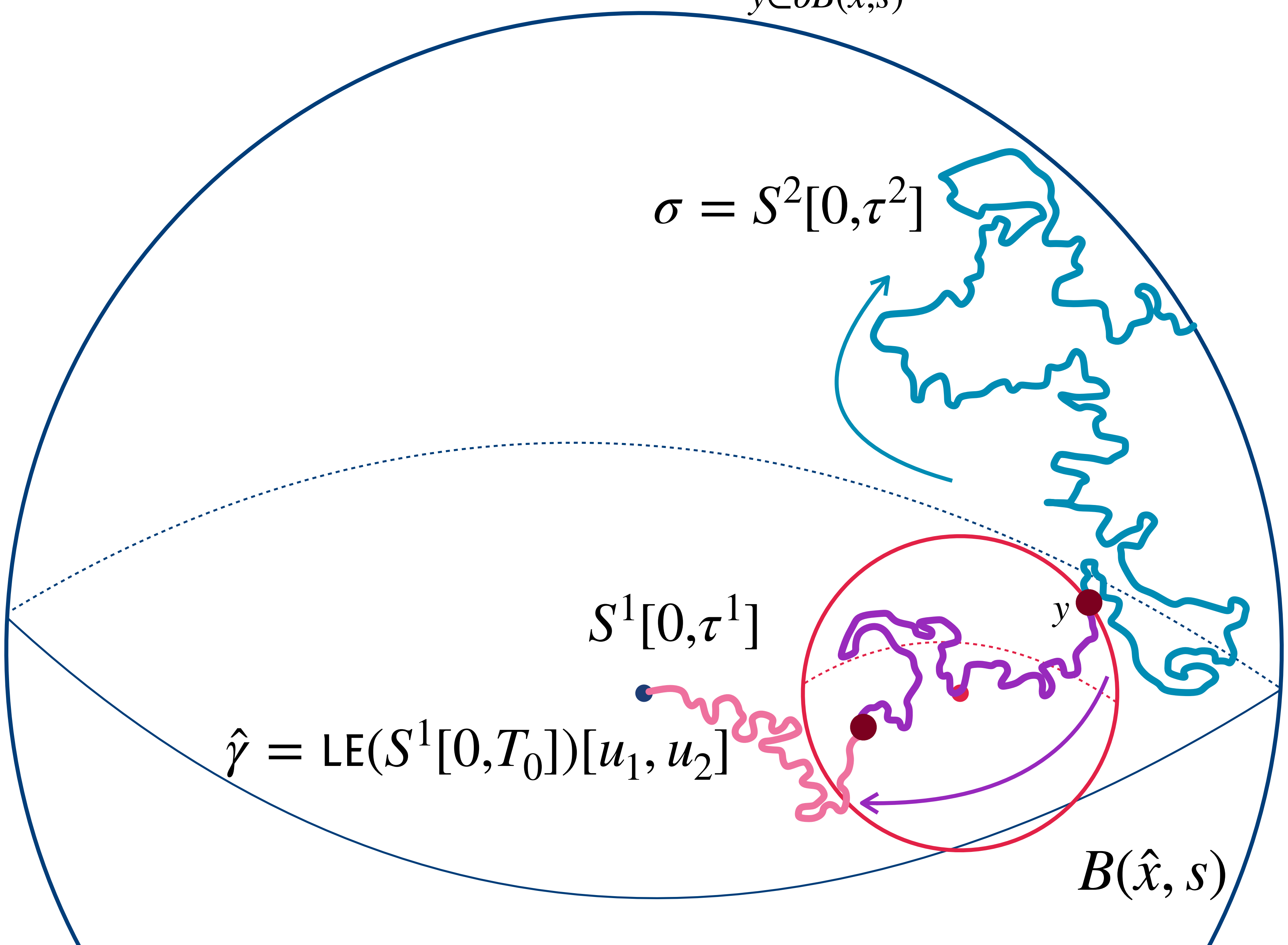
$$\sum_{y \in \partial B(\hat{x},s)} P^{y,y}\left(t_0^1 < \tau^1, \hat{\gamma} \cap \sigma = \emptyset, \sigma \cap B(\hat{x}, s) = \emptyset\right)$$



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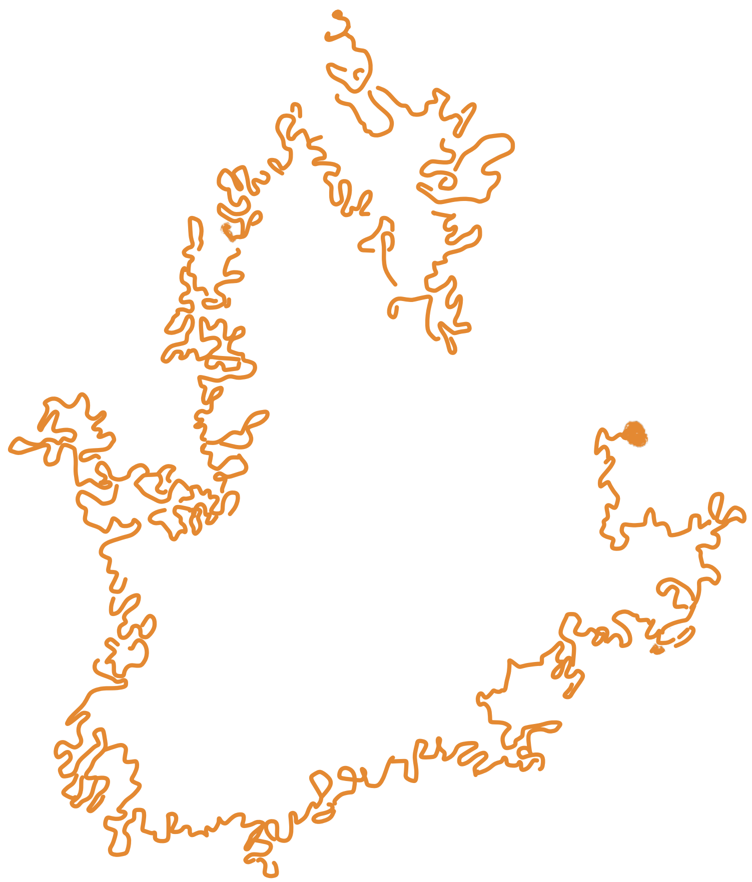
Minkowski content

- $A \subset \mathbb{R}^3$ bounded Borel set



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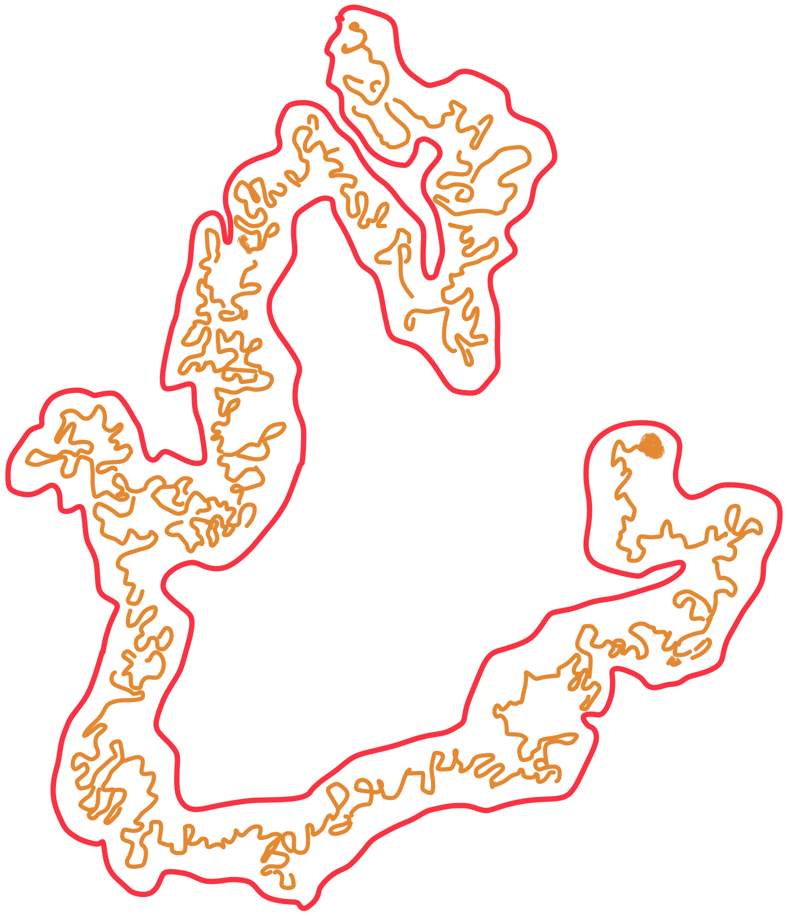

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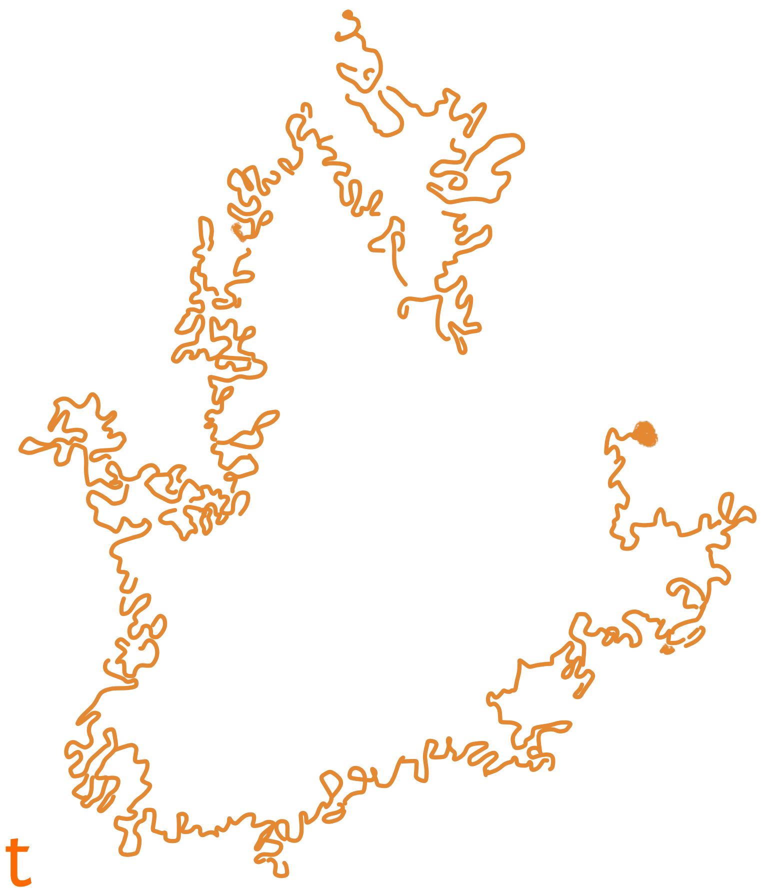

Criterion for existence of Minkowski content

$$g(z) = \lim_{s \to 0} s^{\beta - 3} P\left(B(z, s) \cap \mathscr{K} \neq \varnothing\right)$$

$$g(z, w) = \lim_{s \to 0} s^{2(\beta - 3)} P\left(B(z, s) \cap \mathscr{K} \neq \varnothing, B(w, s) \cap \mathscr{K} \neq \varnothing\right)$$

Minkowski content and limiting occupation measure

[H.T.-Li-Shiraishi, '24]

For any "nice" box $V \subset \mathbb{D}$, the β-Minkowski content

$$\mathrm{Cont}_{\beta}(\mathscr{K} \cap V)$$

exists. Moreover, if

- μ is the limiting occupation measure, and
- ν is measure induced by β-Minkowski content.

There exists a constant $c_0 > 0$

$$\nu = c_0 \mu \quad \text{a.s.}$$

Scaling limit of the 3D LERW

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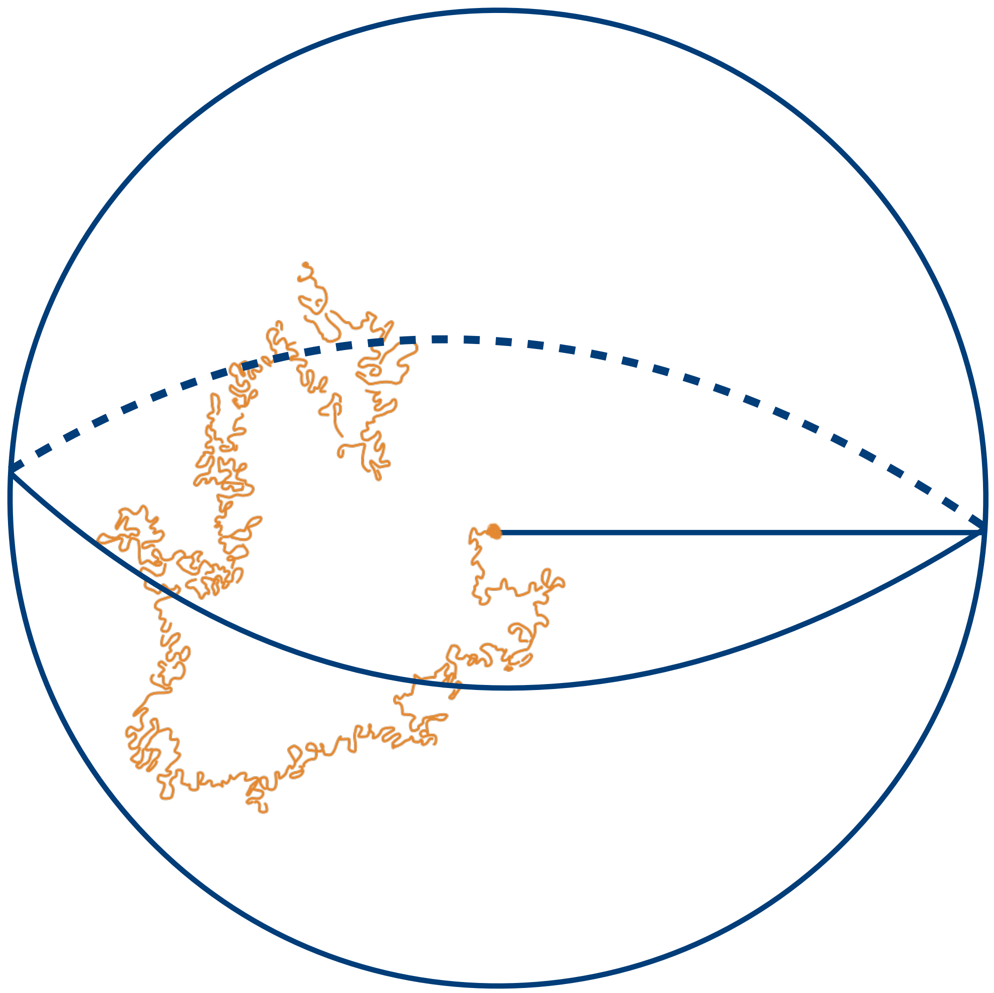

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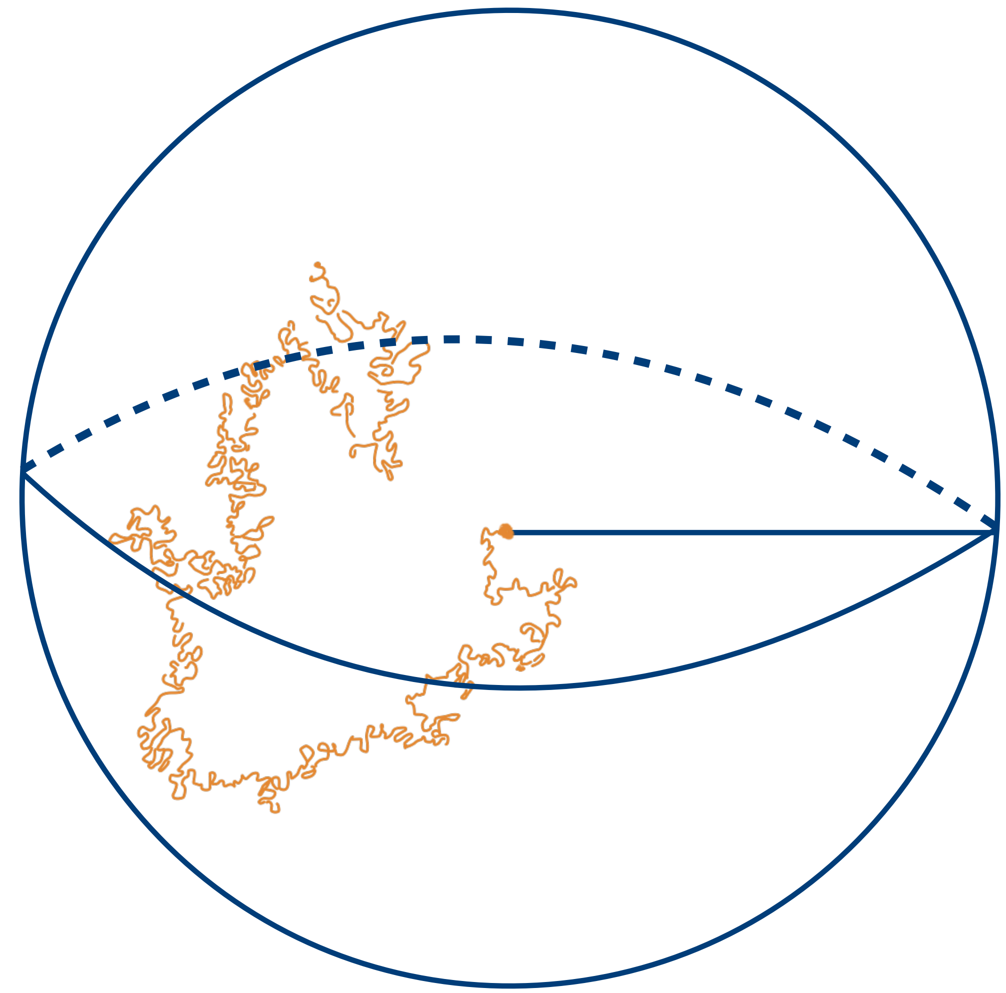

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- Scaling limit of 3D UST [Angel-Croydon-H.T.-Shiraishi, '20]

Thank you!

Sharp one-point estimates and Minkowski content for the scaling limit of three-dimensional loop-erased random walk

Sarai Hernandez-Torres, Xinyi Li, Daisuke Shiraishi

arxiv: 2403.07256